

A Path-Type Weak Factorization System

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- A weak factorization system (WFS) from Awodey-Hua path type semantics.
- The proof of functoriality and clovenness.
- Explicit description of the left and right classes.
- The groupoid model as a case study.

- What is / Why do we care about WFS?
- Our WFS
 - background: Awodey-Hua path type semantics
 - construction: via *path object*
 - properties: functorial + cloven
 - example: the groupoid model
- Future work

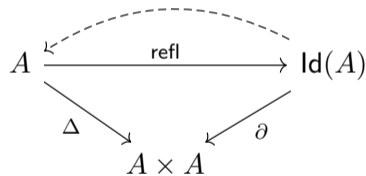
Weak Factorization Systems (WFS)

Why do we care about WFSs?

- Everyone here cares about *intensional* identity types!
- WFSs model intensional identity types soundly and completely.

Extensional

Type families are arrows.

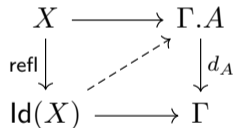


The map refl has a section.

Intensional

← provided by WFS

Type families are fibrations.



Each such square has a diagonal filler.

Definition

A weak factorisation system in a category \mathcal{E} is a couple $(\mathcal{L}, \mathcal{R})$ of classes of morphisms satisfying the following properties:

1. $\mathcal{L} = {}^{\circlearrowleft}\mathcal{R}$ and $\mathcal{R} = \mathcal{L}^{\circlearrowright}$.
2. any morphism $f : X \rightarrow Y$ of \mathcal{E} admits a factorization of the form $f = p \circ i$ with $i \in \mathcal{L}$ and $p \in \mathcal{R}$.

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow i & \nearrow p \\ & E & \end{array}$$

Consequence

Each map i in the *left class* lifts against each map p in the *right class* (notation: $i \pitchfork p$), i.e., for each diagram below, there exists a diagonal filler.

$$\begin{array}{ccc} A & \longrightarrow & E \\ \mathcal{L} \ni i \downarrow & \nearrow & \downarrow p \in \mathcal{R} \\ X & \longrightarrow & B \end{array}$$

- Soundness (Awodey-Warren): WFSs interpret identity types.
← the semantics need not be coherent

$$\begin{array}{ccc} X & \longrightarrow & \Gamma.A \\ \downarrow & \nearrow j & \downarrow \\ \text{Id}(X) & \longrightarrow & \Gamma \end{array}$$

- Completeness (Gambino-Garner): There is a WFS on the syntactic category.
← this WFS is not functorial

Our Case

- Interprets MLTT in a finite limit category.
- Comes out of the HoTTLean project.
- Interprets identity types with coherent substitution.
- Requires only the following minimal assumptions! ← no LCCC!
 - Finite limits.
 - An exponentiable interval, i.e. a bipointed object \mathbb{I} with maps $d_0, d_1 : 1 \rightrightarrows \mathbb{I}$.
- Covers the groupoid model as an example.

Type families are *normal Hurewicz fibrations*.

A *Hurewicz fibration* is a structure $(p : E \rightarrow B, \ell : E \times_B B^{\mathbb{I}} \rightarrow E^{\mathbb{I}})$ as depicted in

$$\begin{array}{ccccc}
 & & & & f^{\mathbb{I}} \\
 & & & & \curvearrowright \\
 E^{\mathbb{I}} & \xleftarrow{\ell} & & & B^{\mathbb{I}} \\
 \searrow^{d_0 \Rightarrow p} & & & & \downarrow \epsilon_0 \\
 & & E \times_B B^{\mathbb{I}} & \longrightarrow & B^{\mathbb{I}} \\
 & & \downarrow \lrcorner & & \downarrow \epsilon_0 \\
 & & E & \xrightarrow{p} & B \\
 \swarrow^{\epsilon_0} & & & & \\
 & & & &
 \end{array}$$

where ℓ is a section of $d_0 \Rightarrow p$.

A Hurewicz fibration is *normal* if the map ℓ satisfies $\rho_E = \ell \langle \rho_{BP}, 1_E \rangle$.

It follows that a normal Hurewicz fibration is a map $p : E \rightarrow B$ that has the *path lifting property with parameters*, in the sense that for every commutative square:

$$\begin{array}{ccc}
 X & \xrightarrow{H'_0} & E \\
 \downarrow i_0 & \nearrow H' & \downarrow p \\
 X \times I & \xrightarrow{H} & B
 \end{array}$$

- $\widetilde{H}' = \ell\langle \widetilde{H}, H'_0 \rangle$ is a diagonal filler;
- H' lifts refl paths to refl paths.

Axiom (1)

There is a normal Hurewicz fibration, called a *universe* $\text{tp} : \mathbb{T}\mathfrak{m} \rightarrow \mathbb{T}\mathfrak{y}$, such that each type family is a pullback of tp .

$$\begin{array}{ccc} \Gamma.A & \longrightarrow & \mathbb{T}\mathfrak{m} \\ \downarrow & \lrcorner & \downarrow \text{tp} \\ \Gamma & \xrightarrow{A} & \mathbb{T}\mathfrak{y} \end{array}$$

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 \end{array}$$

Axiom (2)

There are maps Path and path fitting into the pullback square

$$\begin{array}{ccc}
 \mathbb{T}\mathbf{m}^{\mathbb{I}} & \xrightarrow{\text{Path}} & \mathbb{T}\mathbf{m} \\
 \downarrow \partial \lrcorner & & \downarrow \text{tp} \\
 \mathbb{T}\mathbf{m} \times_{\mathbb{T}\mathbf{y}} \mathbb{T}\mathbf{m} & \xrightarrow{\text{path}} & \mathbb{T}\mathbf{y}
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 \end{array}$$

Axiom (3)

The universe tp has Σ -types.

From the three axioms, for each type X , we obtain a *connection* as a map $X^{\mathbb{I}} \rightarrow X^{\mathbb{I} \times \mathbb{I}}$, assigning each path $p \in X^{\mathbb{I}}$ in the type X a square

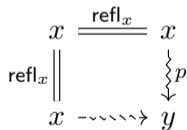
$$\begin{array}{ccc}
 p_0 & \xlongequal{\quad} & p_0 \\
 \parallel & & \downarrow p \\
 p_0 & \dashrightarrow & p_1
 \end{array}$$

The square is a path from refl_{p_0} to p in the path type $\text{Id}(X)$, interpreted by $X^{\mathbb{I}}$.

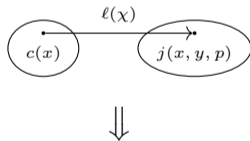
Syntax

$$\frac{\Gamma, x : A, y : A, u : x =_A y \vdash C(x, y, u) \text{ Type} \quad \Gamma, x : A \vdash c(x) : C(x, x, \text{refl}(x))}{\Gamma, x : A, y : A, u : x =_A y \vdash J(x, y, u, c) : C(x, y, u)}$$

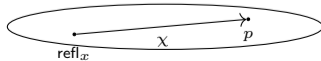
Semantics



χ as a connection



“Total space” C



Path space $A^{\mathbb{I}}$

If

- \mathcal{E} is a finite limit category
- \mathbf{tp} is a universe in \mathcal{E} satisfying Axioms 1–3
- \mathcal{D} is the class of type families classified by \mathbf{tp}
- $\mathcal{C} \hookrightarrow \mathcal{E}$ is the full subcategory spanned by *types* $\{X \mid X \rightarrow 1 \in \mathcal{D}\}$
- \mathcal{L} is the restriction of ${}^{\mathfrak{h}}\mathcal{D}$ to \mathcal{C}
- \mathcal{R} is the restriction of $\mathcal{L}^{\mathfrak{h}}$ to \mathcal{C}

then the pair $(\mathcal{L}, \mathcal{R})$ forms a WFS on \mathcal{C} which

1. coincides with (strong deformation retracts, normal Hurewicz fibrations);
2. has a functorial *path object factorization*;
3. admits an explicit *cleavage* induced by the universal structure on \mathbf{tp} .

$$\begin{array}{ccc} & Y^{\mathbb{I}} & \\ & \downarrow \epsilon_0 & \\ X & \xrightarrow{f} & Y \end{array}$$

$$\begin{array}{ccc} P_f & \longrightarrow & Y^{\mathbb{I}} \\ \downarrow & \lrcorner & \downarrow \epsilon_0 \\ X & \xrightarrow{f} & Y \end{array}$$

- $P_f := X \times_Y Y^{\mathbb{I}}$ is the *path object* of f .

$$\begin{array}{ccc} & Y & \\ & \rho_Y \downarrow & \\ P_f & \longrightarrow & Y^{\mathbb{I}} \\ \downarrow & \lrcorner & \downarrow \epsilon_0 \\ X & \xrightarrow{f} & Y \end{array}$$

- $P_f := X \times_Y Y^{\mathbb{I}}$ is the *path object* of f .

$$\begin{array}{ccc}
 X & \xrightarrow{f} & Y \\
 \downarrow L_f & \lrcorner & \downarrow \rho_Y \\
 P_f & \longrightarrow & Y^{\mathbb{I}} \\
 \downarrow & \lrcorner & \downarrow \epsilon_0 \\
 X & \xrightarrow{f} & Y
 \end{array}$$

- $P_f := X \times_Y Y^{\mathbb{I}}$ is the *path object* of f .
- L_f is a strong deformation retraction.

$$\begin{array}{ccccc}
 X & \xrightarrow{f} & Y & & \\
 \downarrow L_f & \lrcorner & \downarrow \rho_Y & \searrow & \\
 P_f & \xrightarrow{\quad} & Y^{\mathbb{I}} & \xrightarrow{\epsilon_1} & Y \\
 \downarrow & \lrcorner & \downarrow \epsilon_0 & & \\
 X & \xrightarrow{f} & Y & &
 \end{array}$$

- $P_f := X \times_Y Y^{\mathbb{I}}$ is the *path object* of f .
- L_f is a strong deformation retraction.

$$\begin{array}{ccc}
 X & \xrightarrow{f} & Y \\
 \downarrow L_f & \lrcorner & \downarrow \rho_Y \\
 P_f & \xrightarrow{\quad} & Y^{\mathbb{I}} \\
 \downarrow & \lrcorner & \downarrow \epsilon_0 \\
 X & \xrightarrow{f} & Y
 \end{array}
 \begin{array}{c}
 \xrightarrow{\epsilon_1} \\
 \xrightarrow{R_f}
 \end{array}
 Y$$

- $P_f := X \times_Y Y^{\mathbb{I}}$ is the *path object* of f .
- L_f is a strong deformation retraction.
- R_f is a type family.

$$f = R_f \circ L_f$$

$$Y \longrightarrow f \rightarrow X$$

$$Y' \longrightarrow f' \rightarrow X'$$

$$\begin{array}{ccccc} Y & \xrightarrow{f} & X & & \\ & \searrow h & \searrow k & & \\ & & Y' & \xrightarrow{f'} & X' \end{array}$$

$$\begin{array}{ccc}
 Y & \xrightarrow{\quad f \rightarrow \quad} & X \\
 \downarrow \lrcorner & & \downarrow \\
 L_f & & \rho \\
 \downarrow & & \downarrow \\
 P_f & \xrightarrow{\quad \tau \rightarrow \quad} & X^{\mathbb{I}} \\
 \downarrow \lrcorner & & \downarrow \\
 \pi & & \epsilon_0 \\
 \downarrow & & \downarrow \\
 Y & \xrightarrow{\quad f \rightarrow \quad} & X \\
 & \searrow h & \swarrow k \\
 & & Y' \xrightarrow{\quad f' \rightarrow \quad} X'
 \end{array}$$

The diagram illustrates the functoriality of pullbacks. It shows two pullback squares on the left and right, connected by arrows h and k .

- Left Square (Top):** A pullback square with top row $Y \xrightarrow{f} X$, left vertical arrow L_f , right vertical arrow ρ , and bottom row $P_f \xrightarrow{\tau} X^{\mathbb{I}}$. A corner bracket \lrcorner is shown at the top-left.
- Left Square (Bottom):** A pullback square with top row $Y \xrightarrow{f} X$, left vertical arrow π , right vertical arrow ϵ_0 , and bottom row $Y \xrightarrow{f} X$. A corner bracket \lrcorner is shown at the top-left.
- Right Square (Top):** A pullback square with top row $Y' \xrightarrow{f'} X'$, left vertical arrow $L_{f'}$, right vertical arrow ρ , and bottom row $P_{f'} \xrightarrow{\tau} X'^{\mathbb{I}}$. A corner bracket \lrcorner is shown at the top-left.
- Right Square (Bottom):** A pullback square with top row $Y' \xrightarrow{f'} X'$, left vertical arrow τ , right vertical arrow ϵ_0 , and bottom row $Y' \xrightarrow{f'} X'$. A corner bracket \lrcorner is shown at the top-left.

Arrows h and k represent the natural transformations between the pullback functors. h maps the bottom-left Y to the bottom-left Y' , and k maps the bottom-right X to the bottom-right X' .

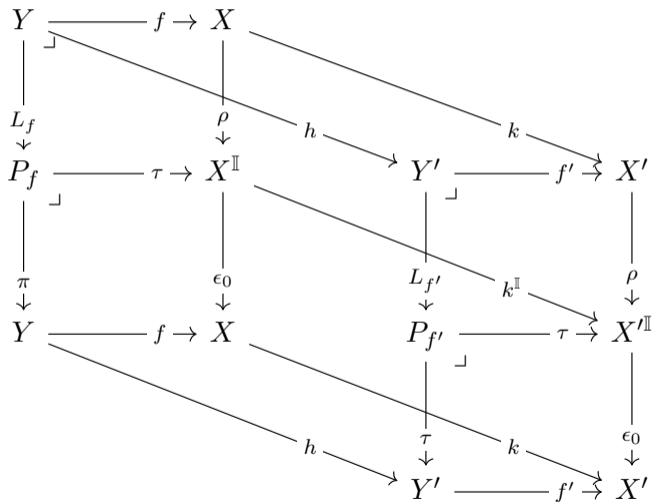
$$\begin{array}{ccccc}
 Y & \xrightarrow{\quad} & f & \rightarrow & X \\
 \downarrow & \lrcorner & & & \downarrow \\
 L_f & & & & \rho \\
 \downarrow & & & & \downarrow \\
 P_f & \xrightarrow{\quad} & \tau & \rightarrow & X^{\mathbb{I}} \\
 \downarrow & \lrcorner & & & \downarrow \\
 \pi & & & & \epsilon_0 \\
 \downarrow & & & & \downarrow \\
 Y & \xrightarrow{\quad} & f & \rightarrow & X \\
 & & & & \downarrow \\
 & & & & \epsilon_0 \\
 & & & & \downarrow \\
 & & & & X' \\
 & & & & \downarrow \\
 & & & & X'^{\mathbb{I}} \\
 & & & & \downarrow \\
 & & & & \epsilon_0 \\
 & & & & \downarrow \\
 & & & & X'
 \end{array}$$

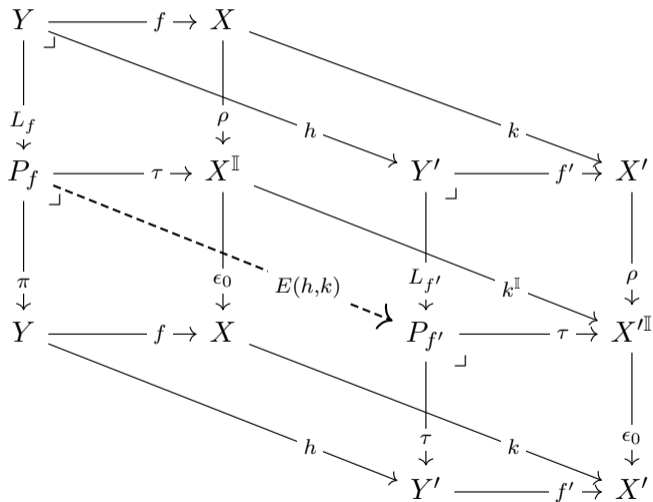
The diagram illustrates the functoriality of pullbacks. It shows a commutative diagram with three rows of objects and several maps between them.

- Top Row:** $Y \xrightarrow{\quad} f \rightarrow X$
- Middle Row:** $P_f \xrightarrow{\quad} \tau \rightarrow X^{\mathbb{I}}$
- Bottom Row:** $Y \xrightarrow{\quad} f \rightarrow X$
- Right Column (Top):** $Y' \xrightarrow{\quad} f' \rightarrow X'$
- Right Column (Middle):** $P_{f'} \xrightarrow{\quad} \tau \rightarrow X'^{\mathbb{I}}$
- Right Column (Bottom):** $Y' \xrightarrow{\quad} f' \rightarrow X'$

The maps and their relationships are as follows:

- Vertical maps: $L_f: Y \rightarrow P_f$, $\rho: X \rightarrow X^{\mathbb{I}}$, $\pi: P_f \rightarrow Y$, $\epsilon_0: X^{\mathbb{I}} \rightarrow X$, $L_{f'}: Y' \rightarrow P_{f'}$, $\rho: X' \rightarrow X'^{\mathbb{I}}$, $\tau: P_{f'} \rightarrow Y'$, $\epsilon_0: X'^{\mathbb{I}} \rightarrow X'$.
- Diagonal maps: $k: X \rightarrow X'$, $k^{\mathbb{I}}: X^{\mathbb{I}} \rightarrow X'^{\mathbb{I}}$, $h: Y \rightarrow Y'$.
- Commutativity: The diagram is commutative, meaning that the composition of maps along any path between two objects is the same.





Central Lemma for Computation (for Clovenness)

Given: A strong deformation retraction
 $(i : X \rightarrow Y, p : Y \rightarrow X, H)$

and a normal Hurewicz fibration
 $(g : E \rightarrow B, \ell : E \times_B B^{\mathbb{I}} \rightarrow P_g)$

$$\begin{array}{ccccc}
 & & X & & \\
 & \nearrow^{i \circ p} & \uparrow H & \searrow & \\
 X & \xrightarrow{i_0} & X \times I & \xleftarrow{i_1} & X
 \end{array}$$

$$\begin{array}{ccccc}
 & & E^{\mathbb{I}} & & \\
 & \searrow^{d_0 \Rightarrow g} & \swarrow^{\ell} & \xrightarrow{g^{\mathbb{I}}} & \\
 E^{\mathbb{I}} & & E \times_B B^{\mathbb{I}} & \longrightarrow & B^{\mathbb{I}} \\
 \downarrow \epsilon_0 & & \downarrow & & \downarrow \epsilon_0 \\
 E & \xrightarrow{g} & B & &
 \end{array}$$

Then: $i \pitchfork g$, with the diagonal filler $j := \epsilon_1 \ell \langle \widetilde{kH}, hg \rangle$

$$\begin{array}{ccc}
 X & \xrightarrow{h} & E \\
 i \downarrow & \nearrow^j & \downarrow g \\
 Y & \xrightarrow{k} & B
 \end{array}$$

A WFS is *cloven* if we have chosen diagonal fillers for each diagram

$$\begin{array}{ccc}
 X & \xrightarrow{L_{L_f}} & P_{L_f} \\
 L_f \downarrow & \nearrow \sigma_f & \downarrow R_{L_f} \\
 P_f & \xlongequal{\quad} & P_f
 \end{array}$$

$$\begin{array}{ccc}
 P_f & \xlongequal{\quad} & P_f \\
 L_{R_f} \downarrow & \nearrow \pi_f & \downarrow R_f \\
 P_{R_f} & \xrightarrow{R_{R_f}} & Y
 \end{array}$$

- Each such diagram is an instance of lifting a left factor against a right factor.
 ← not merely a left map and a right map
- The formula for diagonal fillers applies.

Example: The Groupoid Model

- The universe $\text{tp} : \text{core}(\text{PGrpd}) \rightarrow \text{core}(\text{Grpd})$ acts as $(G, g \in G) \mapsto G$
- The interval \mathbb{I} is the walking isomorphism $\cdot \begin{array}{c} \xrightarrow{\quad} \\ \xleftarrow{\quad} \end{array} \cdot$
- The objects in P_f for a map $f : Y \rightarrow X$ take the form $(y, p : fy \rightarrow x)$
- The factorization is given as

$$\begin{array}{ccc}
 Y & \xrightarrow{f} & X \\
 \searrow & & \nearrow \\
 & P_f & \\
 y \mapsto (y, \text{refl}_{fy}) & & (y, p : fy \rightarrow x) \mapsto x
 \end{array}$$

(injective adjoint equivalences, small cloven isofibrations)

- Our Hurewicz fibrations only lift against 0th endpoint inclusion.

$$\begin{array}{ccc}
 X & \xrightarrow{H'_0} & E \\
 i_0 \downarrow & \nearrow H' & \downarrow p \\
 X \times I & \xrightarrow{H} & B
 \end{array}$$

- They do not have to lift against the 1st endpoint inclusion.
- So our setting is asymmetric.
- Question: Why does this suffice?

Future Direction

Replace Hurewicz fibrations with *unbiased fibrations*.

- Path composition obtained from J-rule is not *definitionally* associative.
← only up to homotopy
- Our WFS is not algebraic.

Future Directions

- In a locally presentable category, every WFS can be made algebraic.
← from Beck's theorem, boring!
- The classifier \mathbf{tp} should determine an algebraic WFS structure.
← hint from Riehl, rather interesting!

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van den Berg + Garner	path object category	functorial
Gambino + Garner	syntactic category	not functorial; shows completeness
Emmenegger	clan + stable path objects	weaker assumptions than our case
Awodey	cofibrant generation	has a full model structure

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Awodey	cofibrant generation	has a full model structure
Our work	interval + universe	determined by the universe

Future Work

- Upgrade to a WFS on the category of polynomials
- Develop examples from cubical type theory
- Compare with biased/unbiased model structure as in Awodey's CCMC
- Formalize in Lean

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