Axioms for higher category theory

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Background

- Homotopy type theory is great
- To do more stuff in HoTT,
  we need a way to deal with ∞-category theory
- Want to extend HoTT in a minimally intrusive way,
  to make this possible

Good news

- The way that ∞-category theory is done today is ontologically compatible with HoTT
- Cisinski, Cnossen, Nguyen, and Walde are currently developing an axiom system that plays well with HoTT

Our contribution

We explain how to extend HoTT with axioms, that should allow for a development of 'all' of higher category theory.[1]

Similar to[2] what CCNW are doing, except

- We use HoTT rather than work 'agnostically'
- Some of their axioms should be provable from the rest, so we omit them
- Foundational issues clarified

[1] This does not include results that rely on taboos like Whitehead's principle or sets cover, unless you postulate those as well.

[2] and inspired by

Setting

We start from book HoTT, with

- Σ
- $=$
- Π with funext
- $\mathbb{N}$
- standard finite types $0, 1, 2$
- a univalent universe U

Types are thought of as spaces / ∞-groupoids / anima
(or objects of some elementary ∞-topos).

Categories

We postulate the following:

- We have a type Cat. Elements of Cat are called *categories*.
- For C, D : Cat, we have a small type $\mathrm{Map}(C, D) : \mathrm{U}$. Elements of $\mathrm{Map}(C, D)$ are called *functors*.
- Given C : Cat, we have $\mathrm{id}_C : \mathrm{Map}(C, C)$. This is called the identity functor.
- Given $F : \mathrm{Map}(C, D)$, $G : \mathrm{Map}(D, E)$, we have $G \circ F : \mathrm{Map}(C, E)$. This is called functor composition.

We also extend HoTT with the following judgments:

- For $F : \mathrm{Map}(A, B)$, we have *judgmental* equality

$$\mathrm{id}_B \circ F \equiv F \equiv F \circ \mathrm{id}_A$$

- For $F : \mathrm{Map}(A, B)$, $G : \mathrm{Map}(B, C)$, $H : \mathrm{Map}(C, D)$, we have

$$H \circ (G \circ F) \equiv (H \circ G) \circ F$$

Coherence for functor composition

Judgmental associativity and unit laws means that an infinite tower of higher coherences (pentagonator etc) are automatic.

We conjecture that it would be sufficient to postulate an axiom schema of weak coherences for functor composition.

But working explicitly with those seems difficult, and intended semantics justifies strict associativity and unit laws.

The form of our postulates

Heuristically, *just adding postulates* is problematic:
what if we are missing higher coherences?

Postulating judgment rules is even worse.

From now on, all[3] postulates are of the form "we have an element of P" where P is a proposition.

(The reason this *ought* to work is that Cat has very few (two) automorphisms.)

[3] With one exception.

Invertible functors

For f : $\mathrm{Map}(C, D)$, the following type is a proposition:

$$\Sigma(g\,h : \mathrm{Map}(D, C)) \times (f \circ g = \mathrm{id}_D) \times (h \circ f = \mathrm{id}_C)$$

and it is equivalent to either one of two conditions

1. For all X : Cat, the map of types $f \circ - : \mathrm{Map}(X, C) \to \mathrm{Map}(X, D)$ is an equivalence
2. For all X : Cat, the map of types $- \circ f : \mathrm{Map}(D, X) \to \mathrm{Map}(C, X)$ is an equivalence

We denote this proposition by $\mathrm{isEquiv}(f) : \mathrm{U}$.

Denote $\Sigma(f : \mathrm{Map}(C, D)) \times \mathrm{isEquiv}(f)$ by $C \simeq D$.

Univalence of Cat

Postulate: for any C : Cat, the following type is contractible.

$$\Sigma(D : \mathsf{Cat}) \times (C \simeq D)$$

Equivalently, we have

$$(C \simeq D) \simeq (C = D)$$

The terminal category

We postulate:

- we have a category $1 : \mathrm{Cat}$,
- for all $C : \mathrm{Cat}$, the type $\mathrm{Map}(C, 1)$ is contractible.

For short, we say ‘Cat has a terminal object’.

Objects of a category

The type $\mathrm{Map}(1, C)$ is denoted by $\mathrm{Ob}(C)$.

Elements of $\mathrm{Ob}(C)$ are called *objects* (of C).

$F : \mathrm{Map}(C, D)$ induces

$$F_{\mathrm{Ob}} : \mathrm{Ob}(C) \to \mathrm{Ob}(D)$$

given by $F_{\mathrm{Ob}}(c) := F \circ c$.

Pullbacks of categories

Given A, B, C : Cat, f : Map(B, A), g : Map(C, A), we postulate:

- a category $B \times_A^{f,g} C$: Cat, or $B \times_A C$ for short
- a functor fst : Map$(B \times_A C, B)$
- a functor snd : Map$(B \times_A C, C)$
- an identification $\alpha : f \circ \mathrm{fst} = g \circ \mathrm{snd}$
- such that for all X : Cat, the map

$$\mathrm{Map}(X, B \times_A C) \to \Sigma(b : \mathrm{Map}(X, B))(c : \mathrm{Map}(X, C))(f \circ b = g \circ c)$$
$$p \mapsto (\mathrm{fst} \circ p, \mathrm{snd} \circ p, \mathrm{ap}_{-\circ p}(\alpha))$$

  is an equivalence of types.

For short, we say that 'Cat has pullbacks'.

Product categories

In particular Cat has binary products $A \times B$, given by $A \times_1 B$.

Note $\mathrm{Ob}(A \times B) \simeq \mathrm{Ob}(A) \times \mathrm{Ob}(B)$.

Functor categories

Given A, B : Cat, we postulate:

- a category $\mathrm{Fun}(A, B)$: Cat
- a functor $\mathrm{ev} : \mathrm{Map}(\mathrm{Fun}(A, B) \times A, B)$
- such that for all X : Cat, the map

$$\begin{aligned} \mathrm{Map}(X, \mathrm{Fun}(A, B)) &\to \mathrm{Map}(X \times A, B) \\ p &\mapsto \mathrm{ev} \circ (p \times A) \end{aligned}$$

  is an equivalence of types.

For short, we say that 'Cat has exponential objects'.

Exercise: construct equivalence $\mathrm{Ob}(\mathrm{Fun}(A, B)) \simeq \mathrm{Map}(A, B)$.

Morphisms and the inverval

We postulate:

1. A category $\mathbb{I}$: Cat
2. An equivalence $e : \mathrm{Ob}(\mathbb{I}) \simeq \{0, 1\}$

We denote $e^{-1}(0)$, $e^{-1}(1)$ simply by 0, 1.

A functor $f : \mathrm{Map}(\mathbb{I}, C)$ is called a *morphism*.

The domain of f is $\mathrm{dom}(f) := f_{\mathrm{Ob}}(0)$;
the codomain is $\mathrm{cod}(f) := f_{\mathrm{Ob}}(1)$.

The fibre of $(\mathrm{dom}, \mathrm{cod}) : \mathrm{Map}(\mathbb{I}, C) \to \mathrm{Ob}(C) \times \mathrm{Ob}(C)$ over (x, y) is denoted $C(x, y)$.

Exercise: $F : \mathrm{Map}(C, D)$ induces $F_{\mathrm{mor}} : C(x, y) \to D(F_{\mathrm{Ob}} x, F_{\mathrm{Ob}} y)$.
For $x : \mathrm{Ob}(C)$ we have $\mathrm{id}_x : C(x, x)$.

Functors into $\mathbb{I}$

$\mathrm{Ob}(\mathbb{I})$ has a linear order where $0 \leq 1$.

Say a function $p : \mathrm{Ob}(C) \to \mathrm{Ob}(\mathbb{I})$ is *monotone* if for all $f : \mathrm{Map}(\mathbb{I}, C)$, we have $p(\mathrm{dom}(f)) \leq p(\mathrm{cod}(f))$.

We postulate:

- For all C : Cat, the function

$$\begin{gathered} \mathrm{Map}(C, \mathbb{I}) \to \mathrm{Ob}(\mathbb{I})^{\mathrm{Ob}(C)} \\ F \mapsto F_{\mathrm{Ob}} \end{gathered}$$

  is (-1)-truncated, and its image consists precisely of monotone maps.

The category $[n]$

We can now prove:
for every n, we have a category $[n]$: Cat, with

1. an equivalence $\alpha : \mathrm{Ob}([n]) \simeq \{0 \ldots n-1\}$, such that
2. for $x, y : \mathrm{Ob}([n])$, $[n](x, y)$ is a proposition, and equivalent to $\alpha(x) \leq \alpha(y)$
3. for all C : Cat, the map

$$\mathrm{Map}(C, [n]) \to \mathrm{Ob}([n])^{\mathrm{Ob}(C)}$$

   is (-1)-truncated, and its image consists precisely of monotone maps

In fact $[n+1] \simeq \mathrm{Fun}([n], \mathbb{I})$.

The Segal axiom

We postulate: the following square of categories is a pushout.

$$\begin{array}{ccc} 1 & \xrightarrow{0} & \mathbb{I} \\ {\scriptstyle 1}\big\downarrow & & \big\downarrow{\scriptstyle 12} \\ \mathbb{I} & \xrightarrow[01]{} & [2] \end{array}$$

Explicitly, this means that for all C : Cat, the canonical map

$$\mathsf{Map}([2], C) \to \mathsf{Map}(\mathbb{I}, C) \times^{\mathsf{cod},\mathsf{dom}}_{\mathsf{Map}(1,C)} \mathsf{Map}(\mathbb{I}, C)$$

is an equivalence of types.

This lets us define composition as an operation

$$\circ : C(y, z) \times C(x, y) \to C(x, z).$$

$[n]$ is the colimit of its spine

We can now prove: for every $C : \mathrm{Cat}$, $n : \mathbb{N}$, 'the map'

$$\mathrm{Map}([n], C) \to \Sigma(a : \mathrm{Ob}(C)^{\mathrm{Fin}(n+1)}) \times \Pi_{i:\mathrm{Fin}(n)} C(a(i), a(i+1))$$

is an equivalence of types.

In other words:

- $[n]$ is freely generated by a sequence of n composable maps
- $[n]$ is the colimit of $\mathbb{I} \xleftarrow{1} 1 \xrightarrow{0} \mathbb{I} \xleftarrow{1} \cdots \xrightarrow{0} \mathbb{I}$

This can be used to prove associativity of composition, etc, by reducing to the universal case ([3] in the case of associativity).

More axioms

We further postulate

- The Rezk axiom: every category is univalent.
- $\mathbb{I} \times \mathbb{I}$ is the pushout of $[2] \leftarrow \mathbb{I} \rightarrow [2]$.
- $\mathbb{I}$ detects equivalences.
- Full subcategories exist.
- For X : U, Cat has X-indexed coproducts.
- Type-indexed coproducts satisfy descent.
- The coproduct $\bigsqcup_X 1$ is a groupoid
- The map $X \to \mathrm{Ob}(\bigsqcup_X 1)$ is an equivalence.
- Cat has pushouts.
- Cat has 'enough' subuniversal left fibrations.

Formalisation

With Johannes Glossner and Jonas Höfer,
we have recently started to formalise some basics in Agda:

`codeberg.org/dwarn/axcat`

This uses some tricks to make sure all CCC laws hold definitionally in Cat.

Conclusion

The exact collection of axioms listed here is not so important.
What is important is:

- The axioms are fairly easy to state and have clear meaning.
- The list of axioms is reasonably short.
- All axioms are propositions, except for enumeration of $\mathrm{Ob}(\mathbb{I})$.
- We fruitfully leverage HoTT in its standard interpretation.
- Developing higher category theory directly from these axioms seems feasible, *even in existing proof assistant (Agda, Rocq)*!

References

[1] Denis-Charles Cisinski et al. *Synthetic Category Theory*. 2026. URL: https://drive.google.com/file/d/1lKaq7watGGl3xvjqw9qHjm6SDPFJ2-0o/view.

[2] Rune Haugseng. *Yet another introduction to ∞-categories*. 2025. URL: https://runegha.folk.ntnu.no/naivecat_web.pdf.

[3] Christian Sattler and David Wärn. *A synthetic construction of universal cocartesian fibrations*. 2026. DOI: 10.48550/ARXIV.2603.28688.

[4] Christian Sattler and David Wärn. *Confluent colimits commute with pullbacks, given descent*. 2025. URL: https://dwarn.se/confluent.pdf.