Nominal Type Theory by Nullary Internal Parametricity

Variable binding by nullary parametricity

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KU Leuven

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- The way you write terms in Ltm depends on the used axiomatization

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- Not plain DTT
- `Ltm` = closed terms only in a name-free context

Axiomatization 1: pairs

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f (lam <n,t>) = n   -- breaks α-equivalence
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Our solution to both issues:
Use a type theory with *nullary internal parametricity*

Internal (binary) parametric type theory?

Cubical Type Theory

- Paths: $I \to A$, with endpoints eqs

(Binary) Parametric TT

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This work: Cavallo Harper theory (interval based) at arity 0

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$$\mathsf{extent} : \quad ((@\mathsf{N} \multimap \mathsf{A}) \to (@\mathsf{N} \multimap \mathsf{B})) \quad \simeq \quad (@\mathsf{N} \multimap (\mathsf{A} \to \mathsf{B}))$$

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Core idea *(observation existed in the community):*

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Summary

$\mathbf{PNTT}$ = nullary CH theory + Nm and name induction + nominal data types

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Some instances:

extent : $((@\mathsf{N} \multimap \mathsf{A}) \to (@\mathsf{N} \multimap \mathsf{B})) \simeq (@\mathsf{N} \multimap (\mathsf{A} \to \mathsf{B}))$
Gel : $\mathsf{Type} \simeq (@\mathsf{N} \multimap \mathsf{Type})$
SAP-Nm : $1 + \mathsf{Nm} \simeq (@\mathsf{N} \multimap \mathsf{Nm})$ -- new

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Also: $\mathsf{Ltm}_1 \simeq (@\mathsf{N} \multimap \mathsf{Ltm})$, where

data Ltm_1 : Type **where**
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Summary

$[-]_0$ traverses the syntax, *except* on Nm, where $[\mathsf{Nm}]_0 = 1 + \mathsf{Nm}$

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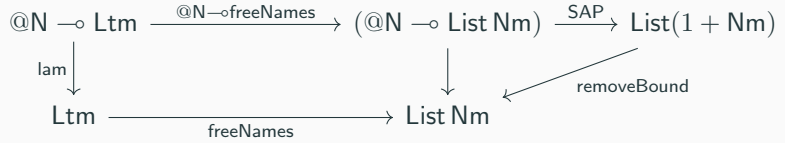

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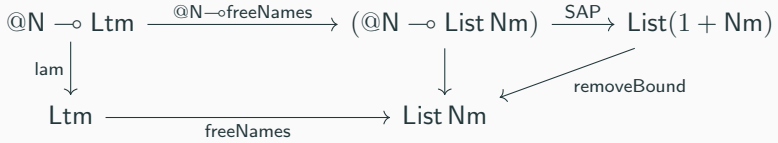

Thus PNTT legitimate nominal framework to study languages with binders

(Contrib.: Synthetic Kripke parametricity)

Nullary parametricity can emulate a type of **"future worlds"** reasoning.

See paper.

Future perspectives: observational PNTT

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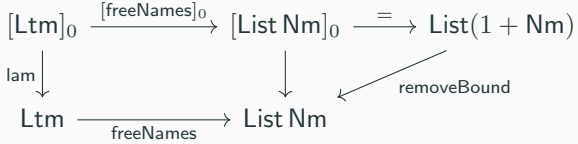

Future perspectives: assessing PNTT

PNTT has untyped names. Is it possible to represent e.g. the syntax of DTT in a good way still?