

Cohomology in realizability models of HoTT

Andrew W Swan

FMF, Univerza v Ljubljani

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Definition

Let \mathcal{E} be a Grothendieck ∞ -topos, X an object of \mathcal{E} and A an internal abelian group in \mathcal{E} . Write $K(A, n)$ for the n th Eilenberg–MacLane space for A . Then we define the n th cohomology group of X with coefficients in A , as

$$H^n(X; A) := \pi_0(\mathrm{Hom}_{\mathcal{E}}(X, K(A, n)))$$

Example

When \mathcal{E} is spaces, this recovers the classical definition of cohomology of a space with abelian group coefficient.

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When \mathcal{E} is $\text{Sh}(T)$, the ∞ -category of sheaves on a topological space and $X = 1$ this recovers sheaf cohomology for the space T . We now allow A to be an abelian group object in $\text{Sh}(T)$ i.e. any sheaf of abelian groups on T .

Example

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Often it useful to work in a slice category of another topos. For example, equivariant cohomology uses \mathcal{E}/BG for a group G .

For cohomology based on Grothendieck ∞ -toposes, it would not typically be interesting or useful to work in \mathcal{E}/N when N is the natural number object of the topos:

Since they are cocomplete, the coproduct $\coprod_{\mathbb{N}} 1$ exists and is a natural number object. Hence $\mathcal{E}/N = \prod_{\mathbb{N}} \mathcal{E}$. The cohomology groups are then simply

$$H^n((X_m)_{m \in \mathbb{N}}; (A_m)_{m \in \mathbb{N}}) \cong \prod_{m \in \mathbb{N}} H^n(X_m; A_m)$$

One can show (e.g. in HoTT) that $K(A, 1)$ is equivalent to the category of A -torsors, and so for spaces X , $H^1(X; A)$ is equivalent to the category of X -indexed families of torsors, which can be stated purely in set theory. Using this formulation Blass found a connection between cohomology and the axiom of choice.

Theorem (Blass)

Work internally in \mathbf{ZF} . For any set $H^1(X; A)$ is trivial for all coefficients A iff X is projective.

We work in an Orton-Pitts model of cubical type theory, so explicitly we have.

- 1 A locally cartesian closed category \mathcal{E}
- 2 An interval object $\delta_0, \delta_1 : 1 \rightrightarrows \mathbb{I}$ in \mathcal{E}
- 3 A class of monomorphisms in \mathcal{E} that we refer to as *cofibrations*.
- 4 Satisfying various axioms.

This gives a model of HoTT including Eilenberg-MacLane spaces $K(G, n)$ as HITs for all abelian groups G and $n : \mathbb{N}$, and including truncations.

Definition

A *homotopy* between maps $f, g : A \rightrightarrows B$ is a morphism $H : A \times \mathbb{I} \rightarrow B$ in the following diagram.

$$\begin{array}{ccc}
 A & & B \\
 \downarrow & \curvearrowright f & \\
 A \times \delta_0 & & \\
 & \searrow & \\
 & A \times \mathbb{I} & \xrightarrow{H} B \\
 & \nearrow & \\
 A \times \delta_1 & & \\
 \uparrow & \curvearrowleft g & \\
 A & &
 \end{array}$$

A *point* of an object A is a map $1 \rightarrow A$.

A *path* between two points is a homotopy between the maps.

A path is *degenerate* if it factors through $\mathbb{I} \rightarrow 1$

Definition

The *connected components functor* $\pi_0 : \mathcal{E} \rightarrow \mathbf{Set}$ is defined by

$$\pi_0(X) := \mathrm{Hom}_{\mathcal{E}}(1, X) / \sim$$

where \sim identifies points if there is a path connecting them.

Definition

Let \mathcal{E} be an Orton-Pitts model. Let X be an object of \mathcal{E} and A a family of internal abelian groups indexed by X .

We define *internal* and *external* cohomology groups as

$$H_{\text{int}}^n(X; A) := \left\| \prod_{x:X} K(A(x), n) \right\|_0$$

$$H^n(X; A) := \pi_0 \left(\left\| \prod_{x:X} K(A(x), n) \right\|_0 \right)$$

The internal cohomology group is an object inside the Orton-Pitts model. The external cohomology group is an actual group in the classical sense.

Our definition is not quite the same as the ∞ -topos definition, but has the following advantage.

Proposition

If $H_{\text{int}}^n(X; A)$ is trivial in the interpretation of HoTT in \mathcal{E} , then $H^n(X; A)$ is the trivial group.

We can also think of this via its contrapositive. If $H^n(X; A)$ is non trivial, then $H_{\text{int}}^n(X; A)$ is not provably trivial in HoTT and so X is not provably projective in HoTT.

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The example will rely on particular features of the model, particularly how the implementation of HITs interact with degenerate and non-degenerate paths. Most arguments just require the following.

- 1 \mathcal{E} is a category of internal presheaves in a locally cartesian closed category.
- 2 The interval is representable.
- 3 Cofibrations are finitely generated as sieves.

Definition (Lumsdaine-Shulman)

Let T be a monad. A *monad cell* on T is a pushout in the category of monads of the following form, where P and Q are free monads on polynomial endofunctors and m is the image of a cofibration.

$$\begin{array}{ccc}
 Q & \xrightarrow{m} & P \\
 \downarrow & & \downarrow \\
 T & \longrightarrow & T'
 \end{array}$$

A *cell monad* is a monad that can be constructed from the identity monad by a finite sequence of monad cells.

A *higher inductive type* is an initial algebra for a cell monad.

Observation

The Coquand-Huber-Mörtberg construction of HITs applies to cell monads.

*This construction is strictly stable under pullback, entirely constructive, and for internal presheaves does not require any choice, including very weak versions such as unbounded regular cardinals or **WISC**.*

Observation

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Note that we have a canonical map from W_0 to W .

Definition

We say a map $A \rightarrow W$ is *point-constructible* if it factors through $W_0 \rightarrow W$.

Theorem

Every point $1 \rightarrow W$ is point-constructible. Hence every degenerate path is point-constructible.

Theorem

If W_0 is a finitary W -type then every map from a representable $yc \rightarrow W$ is either point-constructible or not.

We then switch to a slice of cubical assemblies that does not have decidability of degeneracies. We first make the following definition.

Definition

Let P be a proposition in \mathbf{Asm} . We define \mathbb{I}/P to be the quotient in $\mathbf{Asm}^{\square^{\text{op}}}$ that identifies i and j if either $i = j$ or $\Delta(P)$ is true.

Proposition

For any cubical assembly X , maps $\mathbb{I}/P \rightarrow X$ correspond precisely to paths $p : \mathbb{I} \rightarrow X$ such that P implies p is degenerate.

We use this to define the object B that we will slice over:

$$B := \sum_{e:\mathbb{N}} \mathbb{I}/\varphi_e(e)\uparrow$$

We pullback both \mathbb{N} and \mathbb{Z} , i.e. we are computing the cohomology group $H^1(B^*(\mathbb{N}); B^*(\mathbb{Z}))$ in $\mathbf{Asm}^{\square^{\text{op}}}/B$.

Lemma

We can lift maps $p : \mathbb{I}/P \rightarrow \|X\|_0$ up to homotopy:

$$\begin{array}{ccc}
 2 & \longrightarrow & X \\
 \downarrow & & \downarrow \\
 \mathbb{I}/P & \xrightarrow{p} & \|X\|_0
 \end{array}
 \begin{array}{c}
 \nearrow \\
 \sim \\
 \nearrow
 \end{array}$$

Idea of proof: split into cases depending on whether or not the corresponding path is point constructible.

$$\begin{array}{ccc}
 & & K(\mathbb{Z}, 1)^{\mathbb{N}} \\
 & \nearrow & \downarrow |-\!|_0 \\
 B & \xrightarrow{s} & \|K(\mathbb{Z}, 1)^{\mathbb{N}}\|_0
 \end{array}$$

If we quotient out maps $\mathbb{N} \rightarrow K(\mathbb{Z}, 1)$ directly by homotopy we do get a group. Moreover, we can describe it explicitly as

$$G := \sum_{g: \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{Z}} \varphi_e(e) \uparrow \rightarrow \prod_{n: \mathbb{N}} g(e, n) = 0$$

However, in order to match our definition of cohomology group, and ensure that non triviality implies the internal cohomology is non trivial, we need to quotient further.

Lemma

Suppose that P is a proposition and we have a fibration $Y \twoheadrightarrow \mathbb{I}/P$. Then we can find a section of $\|Y\|_{-1} \twoheadrightarrow \mathbb{I}/P$ iff we can find an element of the disjoint sum $\neg P + \prod_{z:\mathbb{I}/P} Y(z)$.

The composition $| - |_0 \circ t : B \rightarrow \|K(\mathbb{Z}, 1)^{\mathbb{N}}\|_0$ is trivial precisely when we have a section of the truncated type $\|t = 0\|_{-1}$.

Using the above lemma, this happens precisely when there is a computable function $f : \mathbb{N} \rightarrow 2$ such that

- 1 If $f(e) = 0$ then $\varphi_e(e) \downarrow$.
- 2 If $f(e) = 1$ then for all $n : \mathbb{N}$ we have $g(e, n) = 0$.

Theorem

$H^1(B^*\mathbb{N}, B^*(Z))$ has the following group theoretic properties

- ① *It is torsion free*
- ② *It has infinite rank, i.e. \mathbb{Z} -linearly independent subsets of unbounded size.*
- ③ *It embeds infinitely many copies of \mathbb{Q} .*
- ④ *It contains pure elements, i.e. g such that $g \neq nh$ for any $h \in H^1(B^*\mathbb{N}, B^*\mathbb{Z})$ and $n \in \mathbb{Z}$.*
- ⑤ *It cannot be decomposed as $H \oplus L$ where H is c.e. and contains a pure element. In particular it cannot be decomposed as $\mathbb{Z} \oplus H$.*

Lifschitz realizability was originally developed by Lifschitz to separate two versions of Church's thesis: it satisfies Church's thesis in the sense we will use it for this talk "all functions $\mathbb{N} \rightarrow \mathbb{N}$ are computable" but does not satisfy a stronger version of Church's thesis that also applies to multivalued functions.

We can also think of it in terms of reflective subuniverses. It is the largest reflective subuniverse of cubical assemblies that forces **LLPO** to hold.

Definition

Let $m : \mathbb{N}$. Let f be a partial function. For $n : \mathbb{N}$, we say $f(n)$ *avoids* m if whenever $f(n) \downarrow$ we have $f(n) \neq m$.

Theorem

Write φ_e for the Turing machine encoded by $e : \mathbb{N}$. **LLPO** implies that there merely exists $i : 2$ such that $\varphi_e(e)$ avoids i .

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Theorem

There is no (total) computable function $f : \mathbb{N} \rightarrow 2$ such that for all e , $\varphi_e(e)$ avoids $f(e)$.

Proof.

Diagonalisation: if $f = \varphi_e$, then $\varphi_e(e)$ avoids $\varphi_e(e)$. □

Theorem

*Work internally in HoTT and assume Church's thesis and **LLPO**.
Then there is a family of abelian groups $A : \mathbb{N} \rightarrow \mathbf{AbGrp}$ for which $H^1(\mathbb{N}; A)$ is non trivial.*

We define the following two propositions.

$$P_n := \varphi_n(n) \text{ avoids } 0$$

$$Q_n := \varphi_n(n) \text{ avoids } 1$$

We use this to define A_n as a subgroup of $\mathbb{Z}/2\mathbb{Z}$.

$$A_n := \sum_{z:\mathbb{Z}/2\mathbb{Z}} z = 0 \vee (z = 1 \wedge P_n \wedge Q_n)$$

$$X_n := \sum_{z:BA_n} \sum_{p:P_n \rightarrow z=*} \sum_{q:Q_n \rightarrow z=*} \prod_{u:P_n} \prod_{v:Q_n} p(u) \neq q(v)$$

Lemma

Each X_n is contractible.

Proof.

Given an element of P_n , we get a sequence of equivalences:

$$\begin{aligned}
 X_n &\simeq \sum_{z:BA_n} \sum_{p:z=*} \sum_{q:Q_n \rightarrow z=*} \prod_{v:Q_n} \text{refl} \neq q(v) \\
 &\simeq \sum_{q:Q_n \rightarrow **} \prod_{v:Q_n} q(v) \neq \text{refl} \\
 &\simeq Q_n \rightarrow \sum_{q:A_n} q \neq 0 \\
 &\simeq 1
 \end{aligned}$$

The same applies given Q_n , but by **LLPO** we have $P_n \vee Q_n$. □

Lemma

*Let (z, p, q, s) be the unique element of X_n . For each path $z = *$ we can choose an element of $P_n + Q_n$.*

Proof.

We do have $\|P_n + Q_n\|_{-1}$. We just need to choose a witness.

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Given a proof $u : P_n$, we have a loop $p(u)^{-1} \cdot r$ on $*$. We check which element of A_n it corresponds to: either 0 or 1.

In the former case we use $\text{inl}(u)$. In the latter case, recall we only added a non trivial element to A_n if both P_n and Q_n hold, so we can find $v : Q_n$ and use $\text{inr}(v)$ as our witness.

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Similarly given $v : Q_n$, we return $\text{inr}(v)$ if $q(v)^{-1} \cdot r = \text{refl}$ and otherwise we can find $u : P_n$ and return $\text{inl}(u)$. □

Using the contractibility of X_n we can explicitly define $s : \prod_{n:\mathbb{N}} K(A_n, 1)$: it is $\lambda n. \pi_0(x_n)$ where x_n is the unique element of X_n .

From any proof of $s = \lambda n.*$ we get a choice of path $\pi_0(x_n) = *$ for each n . This would give us a choice of $i : 2$ such that $\varphi_n(n)$ avoids i for each n , contradicting Church's thesis.

Hence $s \neq \lambda n.*$. Set truncation preserves inequalities, so furthermore $|s| \neq 0$. We have found a non trivial element of $H^1(\mathbb{N}, A)$. □

- The proof applies more generally to a constructive proof of the special case of Blass' theorem for families of sets with decidable equality and at most two elements.
- We have only shown that the cohomology groups are non trivial. It remains open to say more about their structure.
- Although other models where $\mathbf{AC}_{\mathbb{N},2}$ fails will also have non trivial cohomology groups of \mathbb{N} for the same reason, the precise structure of the group could still vary.

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Thanks for you attention!