Cumulative universes in theory and practice

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Joint work with Jon Sterling

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Outline

1. Introduction
2. Natural universes and The Smallness calculus
3. Equivalence and Normalisation
4. Easy Cumulative Inductive types
5. Conclusion

Introduction

Implementing type theory

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- For different usages, a different tool : Cubical Agda, Rzk...
- We need more and more implementations of type theory, that are often relying on complex higher structures

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- If you are implementing a type theory, you will inevitably have to implement a universe hierarchy
  - → Universes turn types into terms such that types can be used by functions or embedded in inductives
  - → For instance, you need universes to even *state* univalence.
  - → In HoTT, they are often assumed to be cumulative
- But universes are often painful to implement, and lead to many difficult design choices

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- Ultimately, those two presentations should be equivalent, and thus have the same models (See my talk at TYPES'26 !)
- Maybe this is not really about *explicit* and *implicit* syntax...

Codes and injectivity

- The intrinsic difference is the injectivity of the decoding map:

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- Injectivity is a bit weird from a semantics perspective...
  - $\rightarrow$ This looks a bit like a "definitional univalence"
- Yet, injectivity plays an essential role with the realignment property
  - $\rightarrow$ *Chose* coherent codes if the lifts $\uparrow_i^j : \pi_i \rightarrowtail \pi_j$ are monos

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$\rightarrow$ Structural account of "universes à la Coquand"

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  - $\rightarrow$ Semantically compatible with a wide range of models
- Equivalence with the usual *natural* universe hierarchies
  - $\rightarrow$ Normalisation for a theory with **internal level judgements**
- Simpler account of cumulative inductive types
- This has been effectively implemented !
  (https://github.com/raphael-sterbac/elaboration-universes)

Natural universes and The Smallness calculus

Natural universe hierarchies

To have a well-behaved *explicit* universe hierarchy, we need a lot of equations:

$$\mathsf{El}_m(\mathsf{U}_l^m) = \mathsf{U}_l$$

$$\mathsf{El}_l(\Pi^l(a, b)) = \Pi(\mathsf{El}_l(a), \mathsf{El}_l(b))$$

$$\mathsf{El}_m(\uparrow_l^m a) = \mathsf{El}_l(a)$$

(Functoriality)

$$\uparrow_m^n \uparrow_l^m a = \uparrow_l^n a$$

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(Functoriality) *(Naturality)*

$\rightarrow$ We can make this precise in the language of SOGATs, but it gets quite complicated...

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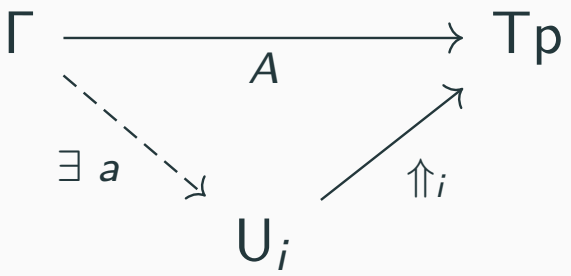

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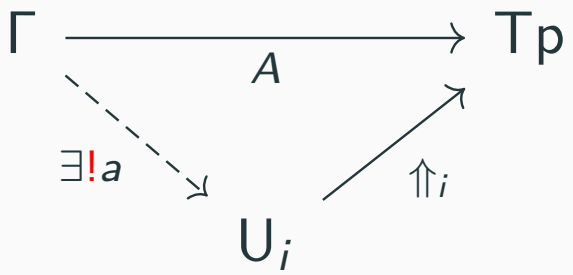

→ This is now a *proposition*, by injectivity of $\Uparrow_i$!

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$$\frac{\Gamma \vdash A \ \mathit{small}_i}{\Gamma \vdash \Downarrow_i A : \mathsf{U}_i} \qquad \frac{\Gamma \vdash A \ \mathit{small}_i}{\Gamma \vdash \Uparrow_i \Downarrow_i A = A \ \mathit{type}}$$

Closure under smallness

Smallness of type formers is now easily stated:

$$\frac{\Gamma \vdash A\ \mathit{small}_i \qquad \Gamma, x : A \vdash B[x]\ \mathit{small}_i}{\Gamma \vdash \Pi(A, B)\ \mathit{small}_i}$$

$$\frac{\Gamma \vdash A\ \mathit{small}_i \qquad \Gamma, x : A \vdash B[x]\ \mathit{small}_i}{\Gamma \vdash \Sigma(A, B)\ \mathit{small}_i} \qquad \frac{\Gamma \vdash i < j}{\Gamma \vdash \mathsf{U}_i\ \mathit{small}_j}$$

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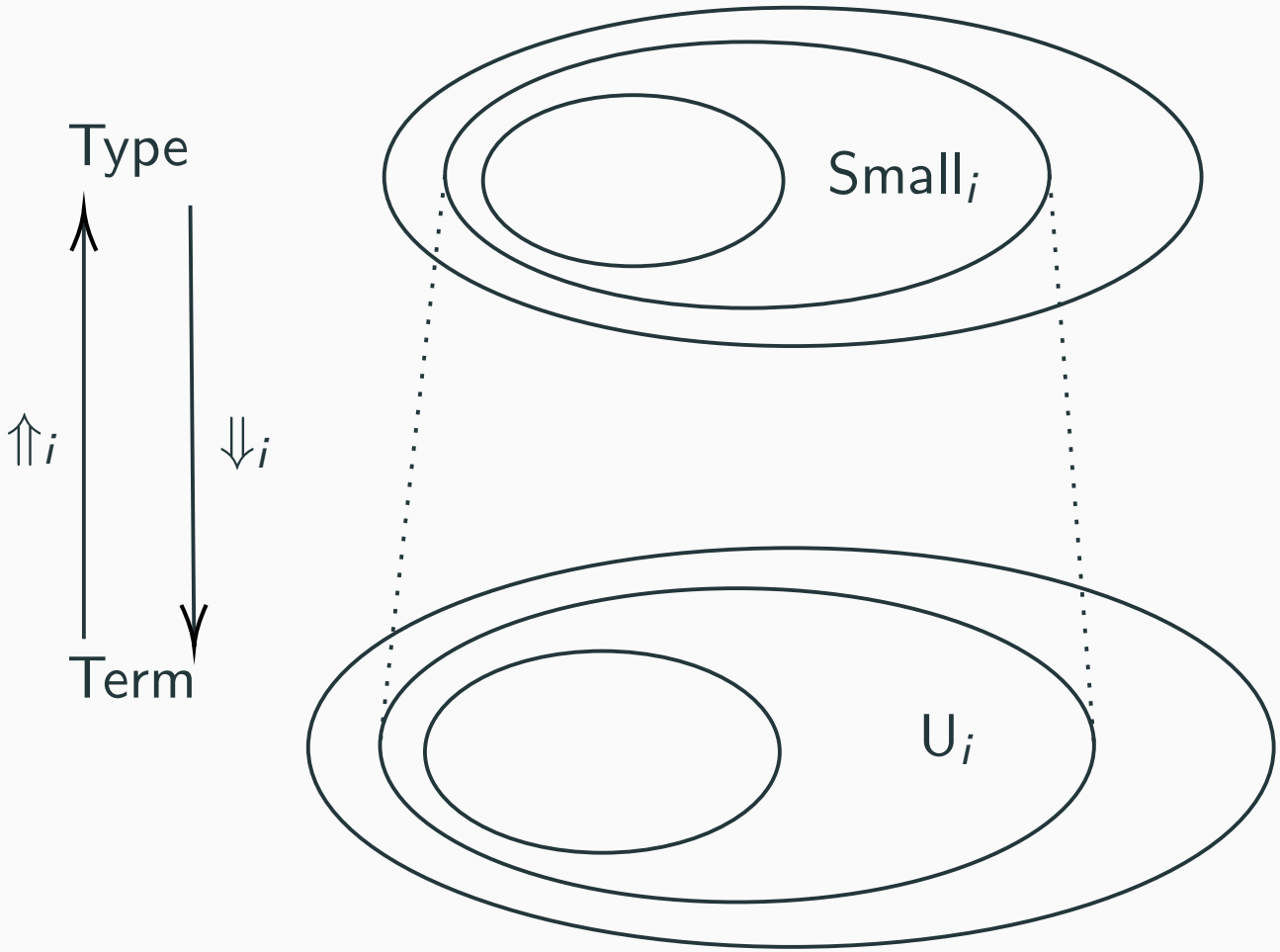
$\Rightarrow$ We can *define* lifts $\uparrow_i^j a := \Downarrow_j \Uparrow_i a$

$\Rightarrow$ We can *define* codes $\Pi^I(a, b) := \Downarrow_i \Pi(\Uparrow_i a, \Uparrow_i b)$ such that:

$$\Uparrow_I \Pi^I(a, b) = \Pi(\Uparrow_I a, \Uparrow_I b)$$
$$\uparrow_I^m \Pi^I(a, b) = \Pi^m(\uparrow_I^m a, \uparrow_I^m b)$$

In short : reflecting types in terms, locally



Working down from the top

- By keeping a top level type judgement, we are able to do *large* things by default, as in standard mathematical practice:

```
theory Category where
    ob: Type
    hom: ob -> ob -> Type
    // operations and axioms...
```

  → No need for universe polymorphism if we don't need small categories !

- Our theory is still compatible with most forms of universe polymorphism
  → We use the one of Bezem, Coquand, Dybjer and Escardo 2024

Equivalence and Normalisation

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$$\mathbb{M}_G.\mathsf{el}_i(M) := \Uparrow_i M$$
$$\mathbb{M}_G.\uparrow_i^j M := \Downarrow_j \Uparrow_i M$$
$$\mathbb{M}_G.\mathsf{u}_i := \Downarrow_i \mathsf{U}_i$$
$$\mathbb{M}_G.\pi_i(M, N) := \Downarrow_i(\Pi(\Uparrow_i M, \lambda x.\, \Uparrow_i N(x)))$$

This constructs F, G morphisms of CwRs for which we can construct a unit and counit.

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- We have internal level judgements: different notion of model
  $\rightarrow$ We cannot directly use standard gluing techniques...
- We use the new *Sconing* technique of Bocqet, Kaposi and Sattler (2023), that applies to any SOGAT !

Normalisation: Structure for universes

We define the computability family for a code $a : \mathsf{U}_i$ as the set of i-small normalisation structures, i.e. of 5-tuples:

$$
\begin{aligned}
a_p &: \mathsf{El}(1, \mathsf{el}_i(a)) \to \mathcal{U}_i && \text{(computability family)} \\
a_N &: [\mathsf{U}_i \ni_{\mathsf{nf}} a] && \text{(normal form)} \\
a_T &: [\mathsf{Tp} \ni_{\mathsf{nf}} \mathsf{el}_i(a)] && \text{(normal form for } \mathsf{el}_i\text{)} \\
a_l &: (j \geq i : \mathsf{Lvl}) \to [\mathsf{U}_j \ni_{\mathsf{nf}} \uparrow_i^j(a)] && \text{(normal form for } \uparrow_i^j\text{)} \\
a_u &: (k : \mathsf{El}(1, \mathsf{el}_i(a))) \to [\mathsf{el}_i(a) \ni_{\mathsf{ne}} k] \to a_p(k) && \text{(unquoting function)} \\
a_q &: (u : \mathsf{El}(1, \mathsf{el}_i(a))) \to a_p \to [\mathsf{el}_i(a) \ni_{\mathsf{nf}} u] && \text{(quoting function)}
\end{aligned}
$$

Now we can prove admissibility of injectivity by induction on the normal forms:

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To summarize, we have the following:

$$\textbf{NatHier} \underset{\text{Admissibility.}}{\Longleftrightarrow} \textbf{NatHierInj} \underset{\text{CwR Equiv.}}{\Longleftrightarrow} \textbf{SmallCalc}$$

Easy Cumulative Inductive types

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$\rightarrow$ The fuss-free hierarchy simplifies this !

Internal datatype descriptions

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$\Rightarrow$ We can define *large* inductives, and work with small ones easily !

An example: Lists

We can define the description of lists by:

$$\mathsf{List}_{\mathsf{D}}(A) = \bigsqcup_{\mathsf{D}}(\{\mathsf{nil}, \mathsf{cons}\}, \{\mathsf{nil} \mapsto \mathsf{unit}_{\mathsf{D}}, \mathsf{cons} \mapsto \bigsqcup_{\mathsf{D}}(A, \lambda x.\, \mathsf{var}_{\mathsf{D}} \times_{\mathsf{D}} \mathsf{unit}_{\mathsf{D}})\}$$

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Which satisfies the smallness predicate only if A is small !

An example: Lists

We can define the description of lists by:

$$\mathsf{List}_{\mathsf{D}}(A) = \bigsqcup_{\mathsf{D}}(\{\mathsf{nil}, \mathsf{cons}\}, \{\mathsf{nil} \mapsto \mathsf{unit}_{\mathsf{D}}, \mathsf{cons} \mapsto \bigsqcup_{\mathsf{D}}(A, \lambda x.\, \mathsf{var}_{\mathsf{D}} \times_{\mathsf{D}} \mathsf{unit}_{\mathsf{D}})\}$$

Then, we recover the well-known signature functor:

$$[\![\mathsf{List}_{\mathsf{D}}(A)]\!] : X \mapsto 1 + A \times X$$

Instanciated on a decoding, we get:

$$[\![\mathsf{List}_{\mathsf{D}}(A)]\!](\mathsf{el}_i(x)) = 1 + A \times \mathsf{el}_i(x)$$

Which satisfies the smallness predicate only if A is small !

Thus we recover :

$$\frac{\_ : \mathsf{Small}_i(A)}{\_ : \mathsf{Small}_i(\mathsf{List}(A))}$$

Conclusion

Summary

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Future work

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Future work

- Shulman showed in 2019 that all ∞-toposes had strict univalent universes, but left cumulativity open
  - $\rightarrow$ It should be possible to use the techniques of Gratzer, Shulman and Sterling (2022) to extend this result !

Thanks for your attention !