Elementary ∞-toposes from type theory

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Categorical models of dependent type theory

Type theory	**Categorical model**
types A, B, ...	objects
$x : A \vdash B(x)$ type	$p : B \twoheadrightarrow A$
$x : A \vdash s(x) : B(x)$	$A \to B$ section of p
$\Pi_{x:A} B(x)$	local exponential $\Pi_A(p)$
$\Sigma_{x:A} B(x)$	composition $B \twoheadrightarrow A \twoheadrightarrow X$
$x : A, y : A \vdash Id_A(x, y)$ type	factorisation $A \to Id_A \twoheadrightarrow A \times A$ of (id, id) ('A is ∞-groupoid')

Various strict frameworks to implement this:

<u>contextual categories</u>, categories with families/attributes, representable map categories, natural models, split comprehension categories...

Internal language conjectures

The previous slide suggests $\left\{\text{Type theories}\right\} \underset{\text{Lang}}{\overset{\mathrm{Cl}_\infty}{\rightleftarrows}} \simeq \left\{\infty\text{-categories}\right\}$

Conjecture (Kapulkin and Lumsdaine 2018): There are ∞-equivalences

$$\begin{array}{ccc}
\mathbf{Mod}_{\mathrm{HoTT}} & \overset{?}{\dashrightarrow} & \mathbf{ElTop}_\infty \\
 & \mathrm{Cl}_\infty^{\mathrm{HoTT}}\,(\simeq) & \\
\downarrow & & \vdots \\
\mathbf{Mod}_{\Pi,\Sigma,\mathrm{Id}} & \longrightarrow & \mathbf{LCCC}_\infty \\
 & \mathrm{Cl}_\infty^{\Pi,\Sigma,\mathrm{Id}}\,(\simeq) & \\
\downarrow & & \downarrow \\
\mathbf{Mod}_{\Sigma,\mathrm{Id}} & \longrightarrow & \mathbf{Lex}_\infty \\
 & \mathrm{Cl}_\infty^{\Sigma,\mathrm{Id}}\ \simeq &
\end{array}$$

construction is goal of this talk
(HoTT: Π, Σ, Id, univalent universes)

functor constructed by Kapulkin 2017

$\simeq$ proved by Kapulkin and Szumiło 2019

Idea: Construct Cl_∞ as $\mathbf{Mod}_{\mathrm{Id},-} \to \mathbf{Tribes} \xrightarrow{\infty\text{-localisation}} \mathbf{Cat}_\infty$ Tribes are proxies for ∞-categories!

Elementary ∞-toposes

Def: An ∞-category C is an *elementary ∞-topos* if

1. C is finitely complete
2. C is locally cartesian closed
3. C has enough object classifiers (univalent morphisms)

What is missing?

Subobject classifier (Rasekh/Shulman) **Finite colimits**

counterexample:
Uemura's model in cubical assemblies

counterexample:
∞-category π-finite spaces (Anel)

Goal: 1. Construct $\mathrm{Cl}_\infty^{\mathrm{HoTT}}$ as $\mathbf{Mod}_{\mathrm{HoTT}} \to \mathbf{UnivTribes} \to \mathbf{ElTop}_\infty$

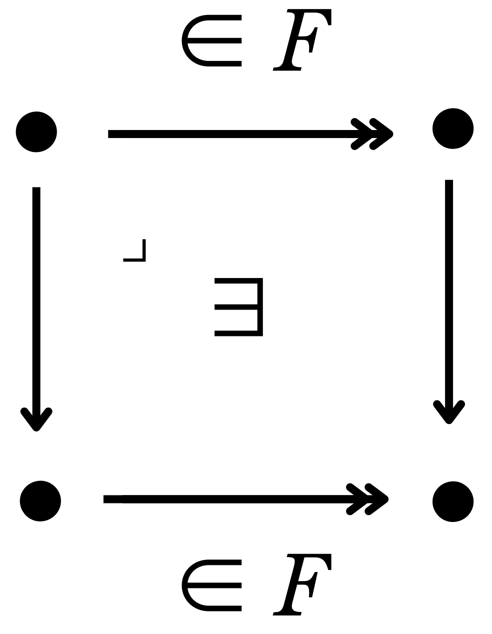
2. propositional resizing ⇒ subobject classifier
pushout types ⇒ finite colimits
in $\mathrm{Cl}_\infty^{\mathrm{HoTT}}(C)$

Tribes

Def (Joyal): A tribe is a category C with a class F of fibrations $\twoheadrightarrow$ such that

(1)

(2)



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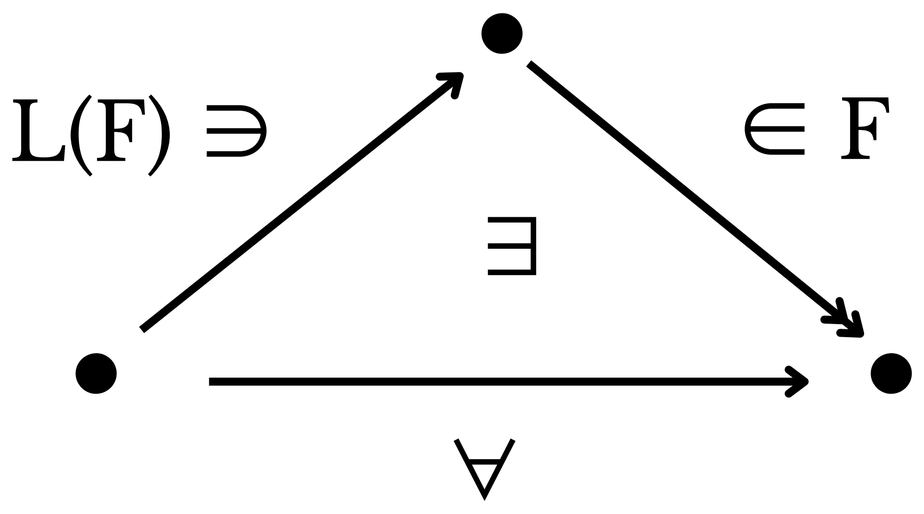

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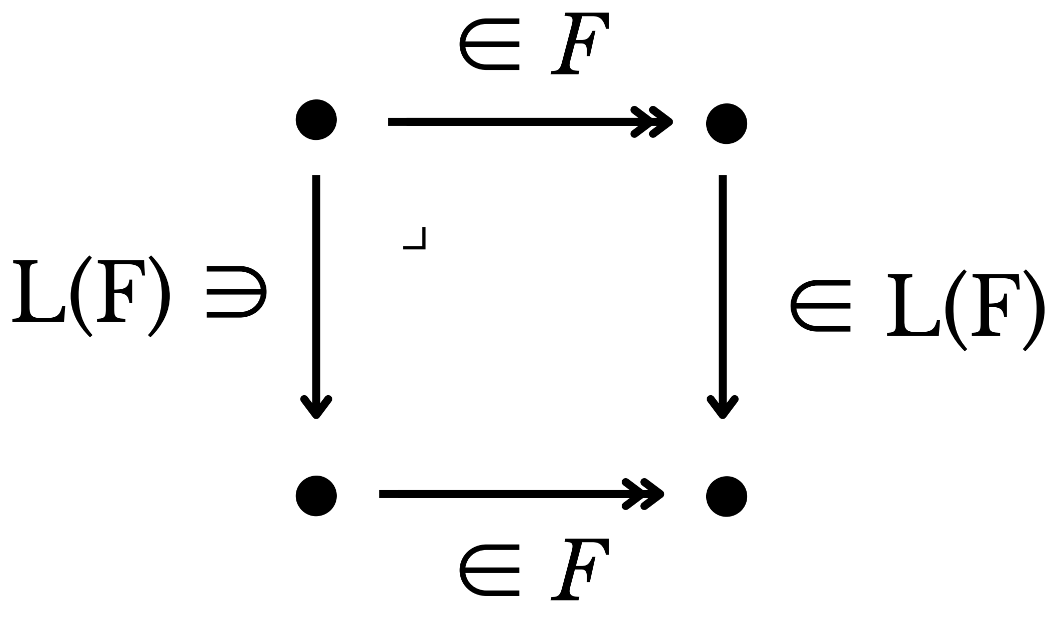
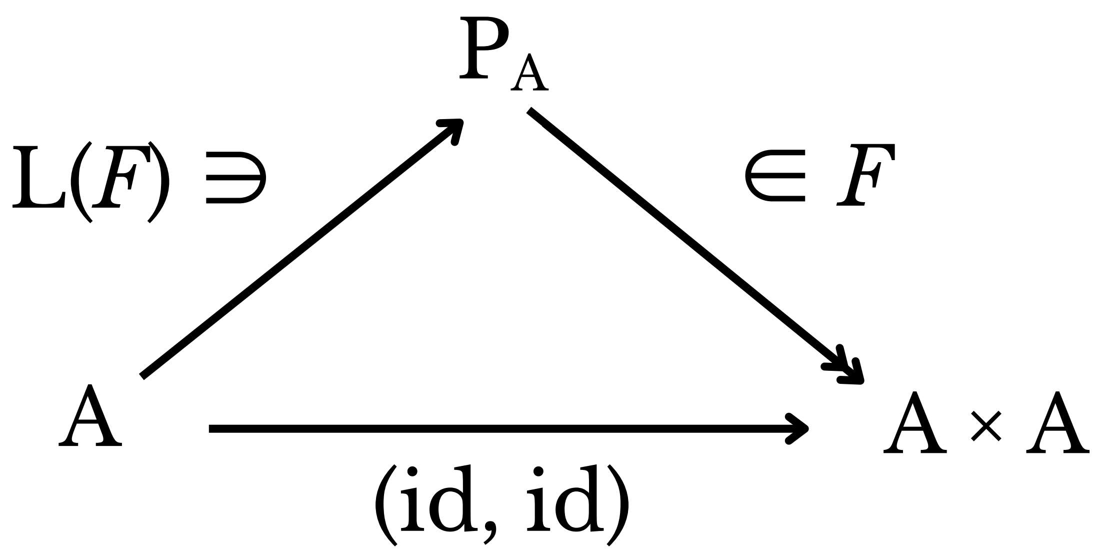


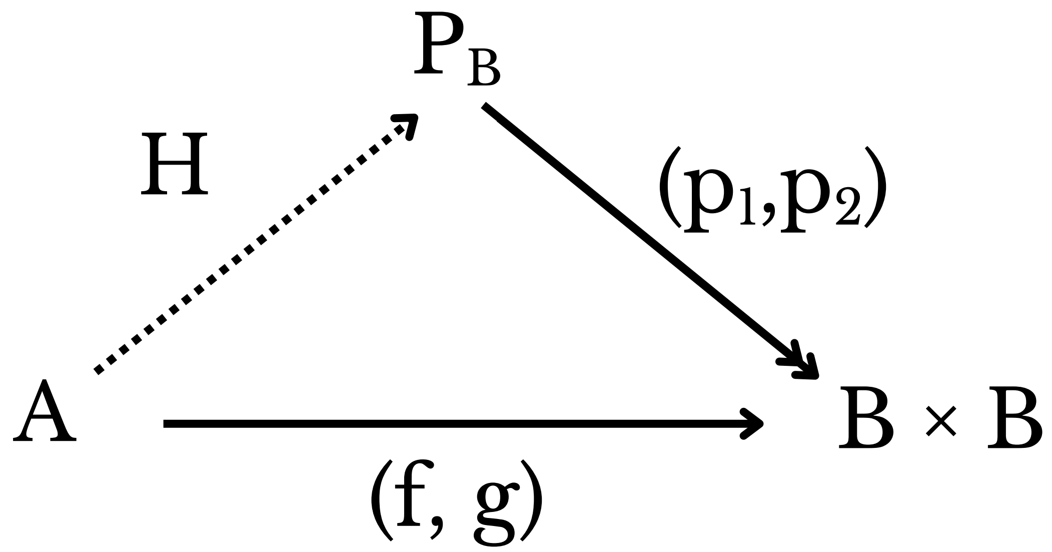
Ex: Kan, Gpd, cat of fibrant obj of right proper model cat (cofibs monos)

Homotopy equivalences in Tribes

Def: A *path object* for an object A in a tribe is a factorisation



Two maps *f* and *g* are *homotopic* ($f \sim g$) if there is H



A map f is a *homotopy equivalence* if there are *g, h* such that $gf \sim \mathrm{id}$ and $fh \sim \mathrm{id}$.

The object of homotopy equivalences

Def: A tribe is a π-*tribe* if pushforwards of fibrations along fibrations exist.

Constr: Let A, B objects. Define $\underline{\mathrm{Eq}}(A, B) := \underline{\mathrm{RInv}}(A, B) \times_{\underline{\mathrm{Hom}}(A, B)} \underline{\mathrm{LInv}}(A, B)$

$$\begin{array}{ccc} \underline{\mathrm{RInv}}(A, B) & \longrightarrow & \underline{\mathrm{Hom}}(B, P_B) \\ \downarrow \quad \lrcorner & & \downarrow (p_{1*}, p_{2*}) \\ \underline{\mathrm{Hom}}(A, B) \times \underline{\mathrm{Hom}}(B, A) & \xrightarrow[(\mathrm{comp},\ \mathrm{const}_{\mathrm{idB}})]{} & \underline{\mathrm{Hom}}(B, B) \times \underline{\mathrm{Hom}}(B, B) \end{array}$$

$$\left\{ X \dashrightarrow \underline{\mathrm{Eq}}(A, B) \right\} \cong \left\{ \begin{array}{ccc} X \times A & \overset{\simeq}{\dashrightarrow} & X \times B \\ & \searrow \; X \; \swarrow & \end{array} \right\} \text{(set of htpy equivalences)}$$

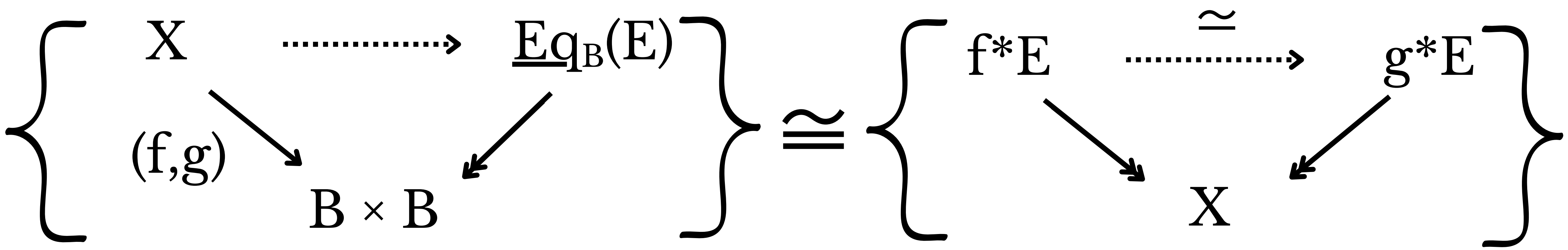
Def: $X \twoheadrightarrow Y$ is a *homotopy monomorphism* if $X \to X \times_Y X$ is homotopy equiv.

Prop: $\underline{\mathrm{Eq}}(A,B) \twoheadrightarrow \underline{\mathrm{Hom}}(A,B)$ is a homotopy mono.

Univalent Tribes

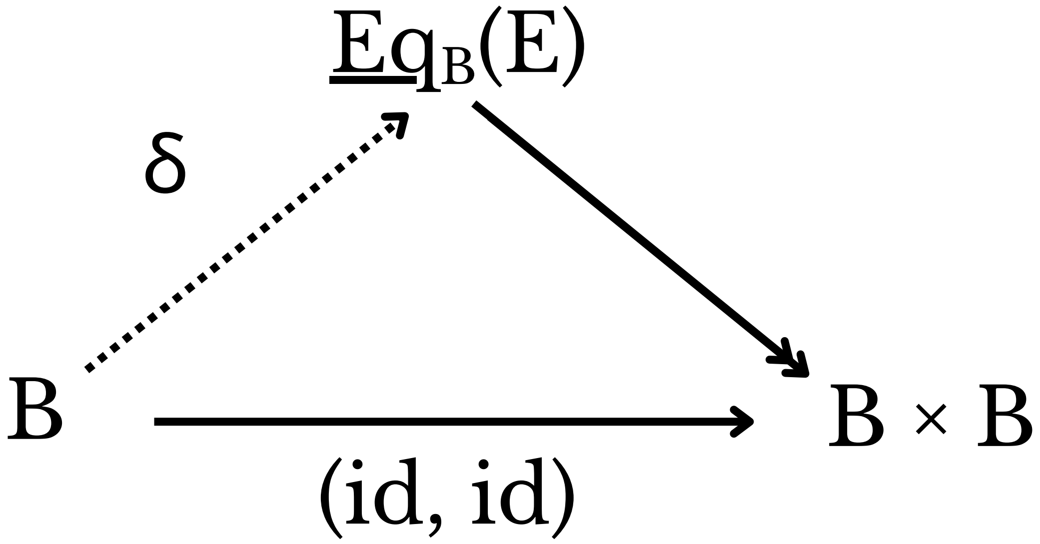
Let $p : E \twoheadrightarrow B$. In the slice $C_{/B \times B}$ define $\underline{Eq}_B(E) := \underline{Eq}(\pi_1{}^*p, \pi_2{}^*p)$ where $\pi_i : B \times B \to B$

Then, for any $f, g : X \to B$



(set of equivalences)

Taking $(f, g) := (id_B, id_B)$ induces δ



Def: A fibration $p : E \twoheadrightarrow B$ is *univalent* if δ is a homotopy equivalence.

Def: A tribe is *univalent* if it has enough univalent fibrations.

Univalence in ∞-categories

Run the analogous constructions in a l.c.c. ∞-category

Prop: $\underline{\mathrm{Eq}}(A,B) \to \underline{\mathrm{Hom}}(A,B)$ is a monomorphism.

Thus, get $\underline{\mathrm{Eq}}_B(E) \to B \times B$ such that

$$\left\{ \begin{array}{ccc} X & \dashrightarrow & \underline{\mathrm{Eq}}_B(E) \\ & {\scriptstyle (f,g)}\searrow \quad \swarrow & \\ & B \times B & \end{array} \right\} \simeq \left\{ \begin{array}{ccc} f^*E & \overset{\simeq}{\dashrightarrow} & g^*E \\ & \searrow \quad \swarrow & \\ & X & \end{array} \right\}$$

(space of equivalences)

(full subspace of mapping space)

Taking $(f, g) := (\mathrm{id}_B, \mathrm{id}_B)$ induces δ

$$\begin{array}{ccc} & \underline{\mathrm{Eq}}_B(E) & \\ {\scriptstyle \delta}\nearrow & & \searrow \\ B & \xrightarrow[(\mathrm{id},\, \mathrm{id})]{} & B \times B \end{array}$$

Def: A morphism $p : E \to B$ is *univalent* if δ is an equivalence.

Applying Cl_∞

Prop: Every model of HoTT is a univalent tribe.

Cl_∞ turns a tribe into an ∞-category inverting all homotopy equivalences.

Thm (Szumiło, Kapulkin, Cisinski): $\mathrm{Cl}_\infty^{\Pi,\Sigma,\mathrm{Id}}$ commutes with dependent products, internal homs, finite (homotopy) limits and slices.

Thus, Cl_∞ preserves everything we needed to define univalence.

Thm: If C is a model of HoTT, then $\mathrm{Cl}_\infty(C)$ is an elementary infinity topos.

Thm: If the type theory has pushouts (HITs) and 0-types then $\mathrm{Cl}_\infty(C)$ has finite colimits.

Subobject classifiers

Thm: For every univalent morphism p there is a subobject classifier $\top_p$ for the monos that are pullbacks of p.

Proof idea: For $p : E \to B$ and $(id,id) : E \to E \times_B E$ using l.c.c. take

$$\Pi_{p\times p}((id,id)) : \Pi_{p\times p}(E) \to B$$

where $p \times p : E \times_B E \to B$. Then, $\top_p$ is the pullback:

$$\begin{array}{ccc} \bullet & \longrightarrow & E \\ {\scriptstyle \top_p}\downarrow & \lrcorner & \downarrow{\scriptstyle p} \\ \Pi_{p\times p}(E) & \longrightarrow & B \end{array}$$

□

Prop: If the type theory satisfies prop. resizing then $Cl_\infty(C)$ has a single subobject classifier.

Future directions

Consider other HITs: quotients, graph colimits, spheres, truncations etc.

Open problem: Show that $\mathrm{Cl}_{\infty}^{\mathrm{HoTT}}$ is an ∞-equivalence

Solving this will depend content of proof that $\mathrm{Cl}_{\infty}^{\Pi,\Sigma,\mathrm{Id}}$ is an ∞-equivalence

Thank you for your attention!