Symmetric products in HoTT

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Overview

1. Classical definition
2. Commutativity in HoTT
3. The definitions of SP^2 and SP^3
4. A sketch of the definition of SP^n

Symmetric products

We define:

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As a (strict) colimit:

$$SP^n X = \operatorname{colim}\left(\; X^n \circlearrowleft^{\sigma_i} \right)$$

In terms of universal property, $SP^n X$ is the space through which every n-argument fully commutative function from X factors uniquely.

The problem with commutativity

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Commutativity structures

A function $f : A^n \to B$ (where $A^n \equiv\colon \mathsf{Fin}\, n \to A$) is commutative in HoTT, if there exists a **commutativity structure**:

$$c : (X : B\Sigma_n) \to (X \to A) \to B, \quad c\ (\mathsf{Fin}\, n) = f,$$

where $B\Sigma_n$ is the type of "unordered" n-element types (realizable as $\sum_{X:\mathsf{Type}} \|X \simeq \mathsf{Fin}\, n\|_{\mathsf{Prop}}$).

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Commutativity comes from paths:

$$p_\sigma : \mathrm{Fin}\, n =_{B\Sigma_n} \mathrm{Fin}\, n, \quad \sigma \in \Sigma_n$$

Borel products

This motivates considering the type:

$$A^n//\Sigma_n = \sum_{X:B\Sigma_n} (X \to A),$$

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called the homotopy quotient (Borel construction). If G acts freely on T, then $T//G \simeq T/G$. However, the action of Σ_n on A^n is not free if it has repeating coordinates:

$$\sigma \cdot (\underbrace{x_1, \ldots, x_1}_{m_1}, \ldots) = (\underbrace{x_1, \ldots, x_1}_{m_1}, \ldots) \quad \text{for} \quad \sigma \in \Sigma_{m_1} \times \cdots \subseteq \Sigma_n$$

A measure of success

We (classically) have a natural map $\Phi : A^n//\Sigma_n \to A^n/\Sigma_n$. For x with all unique coordinates, $\Phi^{-1}\{x\}$ is contractible. In the general case:

$$x = [\underbrace{x_1, \dots, x_1}_{m_1}, \dots] \quad \Rightarrow \quad \Phi^{-1}\{x\} = B(\Sigma_{m_1} \times \dots) = B\Sigma_{m_1} \times \dots$$

To find a space $S^n(A)$ with $\Phi : S^n(A) \to A^n/\Sigma_n$ such that $\Phi^{-1}\{x\}$ is contractible for all x, we need to glue all such subspace in $A^n//\Sigma_n$.

The case $n = 2$

The only subspaces we need to glue are $\Phi^{-1}\{[x, x]\} \simeq B\Sigma_2$. Let $\delta(b, x) = (b, \lambda\_.x)$, then:

$$\begin{array}{ccc} B\Sigma_2 \times X & \xrightarrow{\ \delta\ } & X^2//\Sigma_2 \\ \big\downarrow{\scriptstyle \mathrm{proj}_2} & & \big\downarrow \\ X & \xrightarrow{\qquad} & S^2(X) \end{array}$$

In the classical world, the homotopy pushout $S^2(X)$ is indeed equivalent to SP^2X.

Results about $S^2(X)$

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- Every commutativity structure $c : (X : B\Sigma_2) \to (X \to A) \to B$ with a constant extended diagonal $\delta(X, x) = c(X, \lambda\_.x)$ lifts uniquely to a map from $S^2(X)$.
- By van Kampen, we get that $\pi_1(S^2(A)) = (\pi_1(A))^{ab}$ (which is a baby step towards Hurewicz).

The case $n = 3$

Let's start by gluing $\Phi^{-1}\{[x, x, y]\} \simeq B\Sigma_2$. Let $\delta_2(b, (x, y)) = (b \sqcup *, (\lambda\_.x) \sqcup (\lambda\_.y))$ and:

$$\begin{array}{ccc} B\Sigma_2 \times X^2 & \xrightarrow{\delta_2} & X^3//\Sigma_3 \\ \Big\downarrow{\scriptstyle \mathrm{proj}_2} & \ulcorner & \Big\downarrow \\ X^2 & \longrightarrow & S^3_2(X) \end{array}$$

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$$\begin{array}{ccc} (B\Sigma_3//B\Sigma_2) \times X & \xrightarrow{\delta_3} & S^3_2(X) \\ \downarrow{\scriptstyle \mathrm{proj}_2} & & \downarrow \\ X & \longrightarrow & S^3(X) \end{array}, \quad \begin{cases} \delta_3(\mathrm{in}\ b, x) = \mathrm{inl}(t, \lambda\_.x) \\ ap_{\delta_3(\cdot,x)}(\mathrm{glue}\ b) = \mathrm{glue}(b,(x,x)) \end{cases}$$

The general case

- We glue all subspaces corresponding to single points in SP^nX, going from more general (more refined) to less general (e.g. $[x, y, z, z]$ before $[x, y, y, y]$)
- We denote $S^n_k(X)$ as the partial construction of SP^nX after gluing all subspaces of points with at least k unique coordinates.

Let λ be a possible shape of a point in SP^nX (a partition of n). We reason by induction on the poset of such λ, ordered by refinement. We need to find a type C_λ representing all points that are mapped in Φ to a point of shape λ, or the whole structure of contractions of cases that are refinements of λ.

The general case: one vertex per point

- When gluing subspaces in which several coordinates have the same multiplicity, e.g. $[x, x, y, y]$, we don't want to create each vertex more than once. We will distinguish coordinates with different multiplicities, but identify ones with same multiplicities by using a homotopy quotient, here $X^2//\Sigma_2$

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For λ a shape with n_i coordinates repeating i times (for each i) we define:

$$X^\lambda = (X^{n_1}//\Sigma_{n_1}) \times \cdots \times (X^{n_i}//\Sigma_{n_i}) \times \dots$$

$$X^\lambda_{B\Sigma} = ((X \times B\Sigma_1)^{n_1}//\Sigma_{n_1}) \times \cdots \times ((X \times B\Sigma_{n_i})^{n_i}//\Sigma_{n_i}) \times \dots$$

$$B\Sigma_\lambda(x) \quad \text{s.t.} \quad \sum_{x:X^\lambda} B\Sigma_\lambda(x) = X^\lambda_{B\Sigma}$$

Then $B\Sigma_\lambda(x)$ is merely equal to $(B\Sigma_1)^{n_1} \times \dots$.

The general case: multiple extensions

- We want C_λ to be $X^{\lambda}_{B\Sigma}$ with appropriately glued $C_{\lambda'}$ for λ' refining λ.
- There may be more than one way to extend a point of shape λ to a shape λ', for example $[u, u, v, v]$ to $[a, b, c, c]$ has two. Denote the type of such extensions of x as $T_{\lambda<\lambda'}(x)$, with a function $\mathrm{extend}_{\lambda\to\lambda'}(x, t)$

The general case: the structure to glue

Define $C_\lambda = \sum_{x:X^\lambda} C_\lambda(x)$, where $C_\lambda(x)$ is given by:

- `inX :` $B\Sigma_\lambda(x)$ `->` $C_\lambda(x)$
- `inC : (`λ' `<` λ`) -> (t :` $T_{\lambda'<\lambda}(x)$`) ->` $C_{\lambda'}(\text{extend}_{\lambda\to\lambda'}(x,t))$ `->` $C_\lambda(x)$
- `vertC : (`λ' `<` λ`) -> (t :` $T_{\lambda'<\lambda}(x)$`) ->` $C_\lambda(x)$
- `contrC : (`λ' `<` λ`) -> (t :` $T_{\lambda'<\lambda}(x)$`) (c :` $C'_\lambda(\text{extend}_{\lambda\to\lambda'}(x,t))$`) -> inC` λ' `t c = vertC t`
- `glueC : (`λ' `<` λ`) -> (b :` $B\Sigma_\lambda(x)$`) -> (t :` $T_{\lambda'<\lambda}(x)$`) -> inC` λ' `t (inX (`$\pi_1\,\text{extend}_{\lambda\to\lambda'}(x,t)$`)) = inX b`

The general case: pushout construction

Let $\operatorname{len}\lambda$ be the number of unique coordinates in an element of shape λ. Then $S^n_k(X)$ is the pushout:

$$\begin{array}{ccc} \sum_{\lambda, \operatorname{len}\lambda = k} C_\lambda & \longrightarrow & S^n_{k+1}(X) \\ \downarrow & & \downarrow \\ \sum_{\lambda, \operatorname{len}\lambda = k} X^\lambda & \longrightarrow & S^n_k(X) \end{array},$$

where $S^n_n(X) = X^n // \Sigma_n$. Then $S^n_1 =: SP^n X$.

A desirable application

Let X be a CW-complex based at $*$. Then:

$$SP^\infty X = \operatorname{colim}\left(SP^1 X \hookrightarrow SP^2 X \hookrightarrow SP^3 X \hookrightarrow \dots\right),$$

where $[x_1, x_2, \dots] \mapsto [*, x_1, x_2, \dots]$ This is functorial, $(SP^\infty f)[x_1, \dots] = [f(x_1), \dots]$.

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For $\pi : X \to X/A$ we have:

$$(SP^\infty \pi)^{-1}\{[*, *, \dots, x_1, x_2, \dots]\} = \{[a_1, \dots, a_n, x_1, x_2, \dots] : a_i \in A\} \simeq SP^\infty A$$

So (as long as we prove $SP^\infty \pi$ is a quasifibration, which is trivial in HoTT) there's a fiber sequence $SP^\infty A \to SP^\infty X \to SP^\infty(X/A)$, giving us a homology theory $H_*(X) = \pi_*(SP^\infty X)$.

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- https://github.com/WPaupa/symmetric-products (Arend)

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