Generalized Decidability via Brouwer Trees

Workshop on Homotopy Type Theory / Univalent Foundations
2 June 2026, Aarhus, Denmark

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Decidability

Definition: A proposition P is *decidable* if we can prove $P \vee \neg P$.

Examples:

- Propositions "True" and "False".
- "The integer n is prime".

Semidecidability

Definition [Rosolini 1986, Bauer 2006]: A proposition P is *semidecidable* if there exists $s : \mathbb{N} \to$ Bool such that

$$P \leftrightarrow \exists\,(n : \mathbb{N}).\, s\,n = \mathrm{true}.$$

Examples:

- Decidable propositions.
- Given n, is there a twin prime pair above n, i.e., is there $p > n$ such that p and $p + 2$ are prime?

Beyond semidecidability

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Example:

- Twin Prime Conjecture: There exist infinitely many prime numbers p such that $p + 2$ is also prime.

  This can be formulated as a countable meet of semidecidables:

$$\forall n \, \exists (m > n) \ \mathsf{isPrime}(m) \land \mathsf{isPrime}(\mathsf{m} + 2)$$

Core idea

Can we coherently capture such generalised notions of decidability? Intuitively, this would allow for a more refined view of the "hardness" of a problem, specifying how far it is from being decidable.

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Our approach is to develop a *suitable* theory of ordinals and then use ordinals as a benchmark for weaker notions of decidability.

The type of Brouwer ordinals

Brw : Type is a quotient inductive-inductive type, with constructors [Kraus, Nordvall Forsberg, and Xu, 2023]:

$$\mathsf{zero} : \mathsf{Brw}$$

$$\mathsf{succ} : \mathsf{Brw} \to \mathsf{Brw}$$

$$\mathsf{limit} : (\mathbb{N} \xrightarrow{<} \mathsf{Brw}) \to \mathsf{Brw}$$

$$\mathsf{bisim} : (f, g : \mathbb{N} \xrightarrow{<} \mathsf{Brw}) \to (f \approx g) \to \mathsf{limit}(f) = \mathsf{limit}(g)$$

$$\mathsf{trunc} : (x, y : \mathsf{Brw}) \to (p, q : x = y) \to (p = q)$$

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We need the order relation $(\leq)$ on Brw to define $\xrightarrow{<}$ and $\approx$.

The order relation on Brw

The order relation $\leq$: Brw $\rightarrow$ Brw $\rightarrow$ Prop is defined simultaneously with Brw using:

$$\leq\text{-zero} : \text{zero} \leq x$$

$$\leq\text{-trans} : x \leq y \rightarrow y \leq z \rightarrow x \leq z$$

$$\leq\text{-succ-mono} : x \leq y \rightarrow \text{succ } x \leq \text{succ } y$$

$$\leq\text{-cocone} : x \leq f(k) \rightarrow x \leq \text{limit}(f)$$

$$\leq\text{-limiting} : (\forall k. f(k) \leq x) \rightarrow \text{limit}(f) \leq x$$

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- whether x is finite,
- whether $x = n$ for a fixed finite n,
- whether $x \geq n$ for a fixed finite n,
- whether $x \geq \omega$.

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- Similarly we can show that $x \geq \omega + \omega$ is semidecidable.

Ordinal decidability via Brw

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This motivates the central definition.

Definition: Let $\alpha : \mathsf{Brw}$. A proposition P is α-*decidable* if

$$\exists y : \mathsf{Brw}.\ \big(P \leftrightarrow \alpha \leq y\big).$$

Closure under logical operations: $\wedge$ and $\vee$

A natural question is whether α-decidable propositions are stable under logical operations.

Theorem

For every $\alpha : \mathsf{Brw}$, *if* P *and* Q *are* α*-decidable, then* $P \wedge Q$ *is also* α*-decidable.*

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Theorem

For natural numbers k and n, the $(\omega \cdot k + n)$-decidable propositions are closed under binary disjunction.

So this recovers, and extends, the familiar closure of decidable and semidecidable propositions under $\wedge$ and $\vee$.

Countable meet of semidecidables

Theorem

If $P : \mathbb{N} \to \mathsf{Prop}$ *is semidecidable, then* $\forall n.\, P(n)$ *is* ω^2*-decidable.*

In particular, the twin prime conjecture is ω^2-decidable.

Proof sketch

We need to find a Brouwer ordinal β such that:

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Fix $n : \mathbb{N}$ and assume we have data of semidecidability of P_i for every $i < n$.
Find a *nice* weakly constant function

$$\Psi_n^+ : \Pi(i < n).\, \text{semidec-data}(P_i) \to \text{Brw}$$

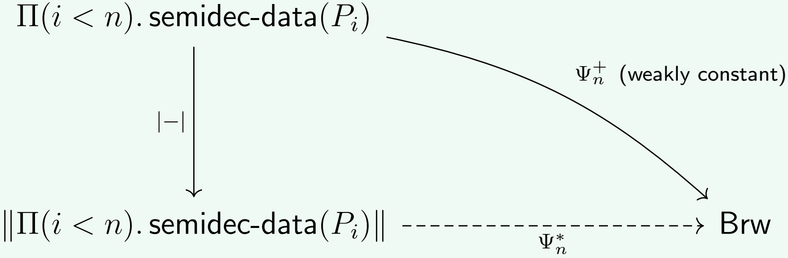
(Recall that a function $f : A \to B$ is weakly constant if $\Pi_{a\,,a':A}\ f\,a = f\,a'$)

Proof sketch

$$\Pi(i < n).\,\mathsf{semidec\text{-}data}(P_i) \xrightarrow{\Psi_n^+ \text{ (weakly constant)}} \mathsf{Brw}$$

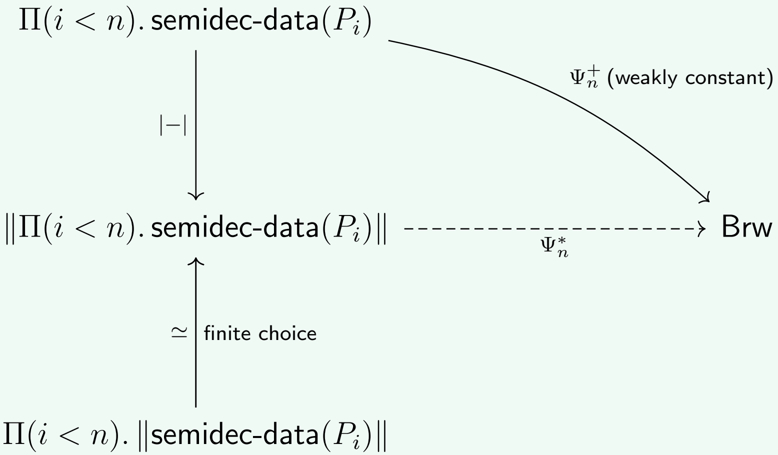
$$\Pi(i < n).\,\mathsf{semidec\text{-}data}(P_i) \xrightarrow{|-|} \|\Pi(i < n).\,\mathsf{semidec\text{-}data}(P_i)\|$$

Proof sketch



Fact: Every weakly constant function to a set can be factored through its domain truncation.

Proof sketch



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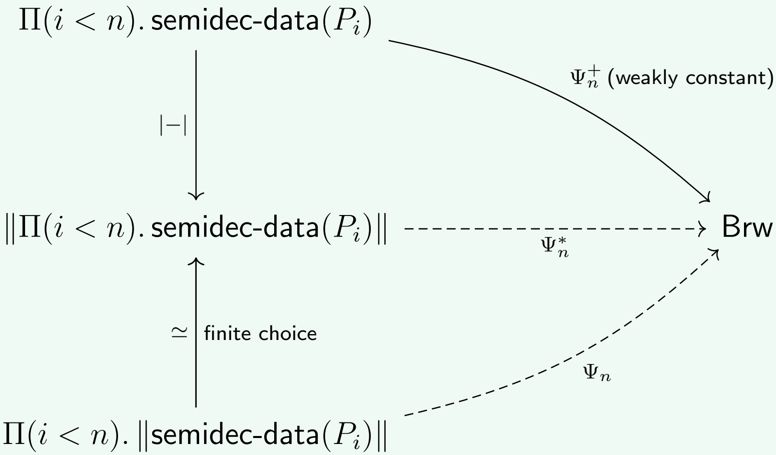

Proof sketch

So for each $n : \mathbb{N}$, we constructed a function

$$\Psi_n : \Pi(i < n).\, \|\mathsf{semidec\text{-}data}(P_i)\| \longrightarrow \mathsf{Brw}$$

with some nice properties (weakly increasing over n, . . .)

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We use this function to define the characteristic ordinal of P , $\Psi(P)$, to be $\textsf{limit}(\lambda n \to \Psi_n + n)$,

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We use this function to define the characteristic ordinal of P , $\Psi(P)$, to be limit$(\lambda n \to \Psi_n + n)$,

and show that $\Psi(P)$ is an evidence that $\forall n.\, P_n$ is ω^2-decidable:

$$\forall n.\, P_n \longleftrightarrow \omega^2 \leq \Psi(P)$$

Countable join of semidecidables

In a constructive setting where countable choice is not assumed, countable join of semidecidable problems is not semidecidable in general, since it implies Escardó-Knapp choice [Escardó, Knapp, 2017].

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If $P : \mathbb{N} \to \text{Prop}$ *is semidecidable, then* $\exists n.\, P(n)$ *is* $(\omega \cdot 3)$*-decidable.*

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Theorem

If $P : \mathbb{N} \to \mathsf{Prop}$ *is semidecidable, then* $\exists n.\, P(n)$ *is* $(\omega \cdot 3)$*-decidable.*

Conversely, every $(\omega \cdot 3)$*-decidable proposition is equivalent to* $\exists m.\, Q(m)$ *for a semidecidable* $Q : \mathbb{N} \to \mathsf{Prop}$*.*

So countable join of semidecidable propositions sit strictly above semidecidability in the hierarchy of ordinal decidability.

Consequences of countable choice

It is well-known that:

Lemma: Assuming countable choice, semidecidable propositions are closed under countable join.

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Lemma: Assuming countable choice, semidecidable propositions are closed under countable join.

From our point of view, this says that $(\omega \cdot 3)$-decidable propositions are $(\omega \cdot 2)$-decidable. We can generalize this:

Theorem

Assuming countable choice, every $(\omega \cdot k)$-decidable proposition is $(\omega \cdot 2)$-decidable (i.e., semidecidable).

Countable choice does not collapse the hierarchy

Theorem

Assume every ω^2-decidable proposition is $(\omega \cdot 2)$-decidable. Then, Markov's Principle implies LPO.

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Assume every ω^2-decidable proposition is $(\omega \cdot 2)$-decidable. Then, Markov's Principle implies LPO.

Remark: There are models where Markov's Principle and countable choice hold, but LPO does not [Hendtlass and Lubarsky, 2016]. Hence countable choice cannot collapse the whole hierarchy, even if it collapses $(\omega \cdot k)$-decidability to $(\omega \cdot 2)$-decidability.

First levels of the hierarchy

Recall: Let α : Brw. A proposition P is α-*decidable* if

$$\exists y : \mathsf{Brw}.\ (P \leftrightarrow \alpha \leq y).$$

$$P \text{ holds} \longleftrightarrow P \text{ is } (\omega \cdot 0)\text{-decidable}$$
$$P \text{ is decidable} \longleftrightarrow P \text{ is } (\omega \cdot 1)\text{-decidable}$$
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P \text{ is countable join of semidecidables} &\longleftrightarrow P \text{ is } (\omega \cdot 3)\text{-decidable} \\
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&\longleftarrow (+ \text{ countable choice})
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Future work

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Other notions of ordinals: Are Brouwer trees the best ordinals to use to categorise decidability? What about Brouwer trees with limits of not just $\mathbb{N}$-indexed sequences?

Summary

We have introduced the notion of α-decidability, for a Brouwer ordinal α.

This generalizes existing notions of decidability and semidecidabilty.

The α-decidable propositions are closed under conjunction, and many other connectives and quantifiers for more restricted α.

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Thank You!

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