Sheaves in HoTT
with applications to synthetic algebraic geometry

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Aarhus

Overview

We present the machinery used to build constructive models for synthetic algebraic geometry and synthetic Stone duality.

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What makes it work?

- We want to study some spaces synthetically.
- Our spaces of interest are build by gluing some basic spaces[1].
- These basic spaces have an algebra/geometry duality.

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- Our spaces of interest are build by gluing some basic spaces[1].
- These basic spaces have an algebra/geometry duality.

What prerequisite to use it?

- You don't need technical ∞-categories know-how.
- You do need familiarity with HoTT and its lex modalities.

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Outline

Synthetic geometry

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Remark

Historical approaches didn't use MLTT.

What do we get?

Slogan

- All types come with a topology/smooth structure/etc.
- All maps are continuous/smooth/etc.

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Remark

This means we have to abandon LEM and AC.

Examples of synthetic geometries

Name	Spaces of interest
Synthetic differential geometry[2]	Smooth manifolds
Synthetic domain theory[3]	Domains
Synthetic topology[4]	Sequential spaces
Synthetic algebraic geometry[5]	Schemes
Condensed type theory[6] / Synthetic Stone duality[7]	Stone spaces, locally compact Hausdorff spaces

[2] [Lawvere 67], [Kock 77], [Dubuc 79], etc.
[3] [Hyland 91], [Fiore, Plotkin 96], etc.
[4] [Escardó 04], [Lešnik 10], etc.
[5] [Bleschmidt 17], [Cherubini, Coquand, Hutzler 24], etc.
[6] [Barton, Commelin 24]
[7] [Cherubini, Coquand, Geerlings, Moeneclaey 24]

Synthetic homotopy theory

We want to study ∞-groupoids (aka homotopy types) synthetically.

Method

1. MLTT can be interpreted in ∞-groupoids.
2. This validates HoTT ($=$ MLTT $+$ univalence $+$ HITs).
3. Work in HoTT.

We can do both!

Method

1. Embed the spaces of interest in an ∞-topos.
2. Interpret HoTT in this ∞-topos.
3. Find new axioms validated by this interpretation.
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The spaces of interest are then sets satisfying some proposition.

Why do this?

- To use some homotopy theory (e.g. shape or cohomology).
- For the convenience of univalence and HITs.

Examples

Name	Spaces of interest
Real-cohesive homotopy type theory[8]	Locally contractible spaces and their homotopy types
Synthetic algebraic geometry[9]	Schemes
Condensed type theory[10]/ Synthetic Stone duality[11]	Stone spaces, locally compact Hausdorff spaces
Simplicial type theory[12]/ Triangulated type theory[13]	Higher categories

[8] [Shulman 18]
[9] [Bleschmidt 17], [Cherubini, Coquand, Hutzler 24], etc.
[10] [Barton, Commelin 24]
[11] [Cherubini, Coquand, Geerlings, Moeneclaey 24]
[12] [Riehl, Shulman 17], etc.
[13] [Gratzer, Weinberger, Buchholtz 24]

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3. Use the corresponding sheaf ∞-topos.

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How to find the new axioms?

Need axioms that are:

- Validated by the interpretation of HoTT into this topos.
- Enough to define and study our spaces of interest.

This is usually very challenging.

Algebra/geometry dualities

Name	Algebra	Geometry
Zariski duality	Rings	Affine schemes
Stone duality	Boolean algebras	Stone spaces
//	Distributive lattices	Coherent spaces
Gelfand duality	C^*-algebras	Compact Hausdorff spaces
⋮	⋮	⋮

Back to our goal

Let's focus on synthetic algebraic geometry.

Goal

We want to work synthetically with schemes.

Basic spaces are affine schemes.

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Leverage the rings/affine schemes duality to get nice axioms validated by the relevant topoi.

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Goal

We want to work synthetically with schemes.

Basic spaces are affine schemes.

Core idea

Leverage the rings/affine schemes duality to get nice axioms validated by the relevant topoi.

- [Kock 80] for external duality used in 1-topoi.
- [Bleschmidt 17] for internal duality in 1-topoi.
- [Coquand, Höfer, Sattler 26] for internal duality in ∞-topoi.

Outline

Goal

This section presents results from [Coquand, Höfer, Sattler 26].

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We assume given $\mathbb{T}$ an algebraic theory.

Idea

$\mathbb{T}$ models the duals to our basic spaces.

For synthetic algebraic geometry, $\mathbb{T}$ is the theory of rings.

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Idea

$\mathbb{T}$ models the duals to our basic spaces.

For synthetic algebraic geometry, $\mathbb{T}$ is the theory of rings.

Intermediary goal

Work synthetically in the ∞-topos of presheaves on $(\mathbb{T}\text{-Alg}_{fp})^{op}$.

Need axioms satisfied by the interpretation of HoTT in this topos.

The generic object

We have a presheaf sending any $A : \mathbb{T}\text{-Alg}_{fp}$ to its underlying set.
It is a $\mathbb{T}$-algebra in $\text{PSh}((\mathbb{T}\text{-Alg}_{fp})^{op})$.

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Axiom 1 (Generic object)

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For synthetic algebraic geometry, we have a ring R.

The spectrum construction

Definition

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Example from synthetic algebraic geometry

Given $P : R[X]$, then

$$\mathrm{Spec}(R[X]/P) = \mathrm{Hom}_{R\text{-Alg}}(R[X]/P, R) = \{x : R \mid P(x) = 0\}.$$

Duality axiom

Axiom 2 (Duality)

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The map

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Definition

Sets in the image of Spec are called affines.

Duality axiom

Corollary

Given $A, B : \mathbb{G}\text{-Alg}_{fp}$, we have that

$$(\mathsf{Spec}(B) \to \mathsf{Spec}(A)) = \mathsf{Hom}_{\mathbb{G}\text{-Alg}}(A, B).$$

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Given $A, B : \mathbb{G}\text{-Alg}_{fp}$, we have that

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Example from synthetic algebraic geometry

$$(R \to R) = \mathrm{Hom}_{R\text{-Alg}}(R[X], R[X]) = R[X]$$

Choice for affines

We know that representables in a presheaf category are projective.

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Axiom 3 (Choice for affines)

Given $A : \mathbb{G}\text{-Alg}_{fp}$, the set $\mathrm{Spec}(A)$ has choice.

Presheaf models summarized

Theorem [Coquand, Höfer, Sattler 26]

Given $\mathbb{T}$ an algebraic theory, there exists a model of HoTT where:

- There is a $\mathbb{T}$-algebra $\mathbb{G}$.
- For all $A : \mathbb{G}\text{-Alg}_{fp}$, the map $A \to \mathbb{G}^{\mathsf{Spec}(A)}$ is an equivalence.
- For all $A : \mathbb{G}\text{-Alg}_{fp}$, the set $\mathsf{Spec}(A)$ has choice.

Outline

Toward sheaves

We want to work with sheaves instead of presheaves.

External approach

Redo [Coquand, Höfer, Sattler 26] with sheaves.

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Internal approach

We define sheaves in (HoTT + presheaf axioms).

Internal topologies

Definition

A topology consists of a class $\mathbb{J}$ of finite sums of affines such that $1 \in \mathbb{J}$ and $\mathbb{J}$ is stable under Σ.

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Given S affine, any $\mathbb{J}$-cover over S is merely of the form

$$U_1 + \cdots + U_n \to S$$

with each U_i affine.

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Lemma

- Isomorphisms are $\mathbb{J}$-cover.
- $\mathbb{J}$-covers are stable under pullbacks and composition.

Internal sheaves

Definiton

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Given a $\mathbb{J}$-cover $T \to S$ and a set X that is a $\mathbb{J}$-sheaf, the map

$$X^S \to \lim(X^T \rightrightarrows X^{T \times_S T})$$

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Remark

There is a version for n-types based on iterated joins [Williams 25].

Internal sheaves

Note that being a $\mathbb{J}$-sheaf is an accessible lex modality.

Corollary [Rijke, Shulman, Spitters 20]

- $\mathbb{J}$-sheaves are stable under Σ and identity types.
- $\mathbb{J}$-sheaves are stable under Π_X for any type X.
- We have a $\mathbb{J}$-sheafification.
- The type of $\mathbb{J}$-sheaves is a itself a $\mathbb{J}$-sheaf.

Lex modalities and submodels of HoTT

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So $\mathbb{J}$-sheaves should form a model of HoTT.

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Theorem [Quirin 16]

Given an accessible lex modality L, we have an interpretation $[\![\_]\!]$ of HoTT into L-modal types where:

$$
\begin{array}{ccc}
[\![\mathrm{Type}]\!] & = & \mathrm{Type}_L \\
[\![\Sigma_X Y]\!] & = & \Sigma_{[\![X]\!]} [\![Y]\!] \\
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So we have a $\mathbb{J}$-sheaf model of HoTT.

The sheaf axioms: generic object

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Lemma (Generic object in sheaves)

In the $\mathbb{J}$-sheaf model, there is a $\mathbb{T}$-algebra $\mathbb{G}$.

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So we have $\mathbb{G}$ in the $\mathbb{J}$-sheaf model. Now we want to have $\mathbb{J}$.

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Definition

A topology is simply defined if:

- $\mathbb{J}$ can be defined from (HoTT + there is $\mathbb{G}$ a $\mathbb{T}$-algebra).
- $[\![X \in \mathbb{J}]\!]_{\mathbb{J}}$ implies $L_{\mathbb{J}}(X \in \mathbb{J})$.

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The sheaf axioms: generic object

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Lemma

In the $\mathbb{J}$-sheaf model, for all $S \in \mathbb{J}$ we have $\|S\|$.

The sheaf axioms: duality

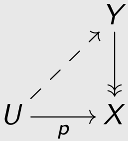
Lemma (Duality in sheaves)

In the $\mathbb{J}$-sheaf model, duality holds.

The sheaf axioms: local choice

Definition

A type X has $\mathbb{J}$-local choice if given a surjection $Y \twoheadrightarrow X$, there exists a type $\mathbb{J}$-cover $p : U \to X$ with a dotted arrow



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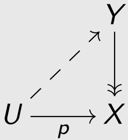

Lemma (Local choice)

In the $\mathbb{J}$-sheaf model, affines have $\mathbb{J}$-local choice.

Sheaf models summarized

Theorem

Assume given:

- $\mathbb{T}$ an algebraic theory
- $\mathbb{J}$ a subcanonical and simply defined topology.

Then there exists a $\mathbb{J}$-sheaf model of HoTT where:

- There is a $\mathbb{T}$-algebra $\mathbb{G}$ such that for all $S \in \mathbb{J}$ we have $\|S\|$.
- For all $A : \mathbb{G}\text{-Alg}_{fp}$, the map $A \to \mathbb{G}^{\mathsf{Spec}(A)}$ is an equivalence.
- For all $A : \mathbb{G}\text{-Alg}_{fp}$, the set $\mathsf{Spec}(A)$ has $\mathbb{J}$-local choice.

Outline

The presheaf model

Proposition

We have a model of HoTT where:

- There is a ring R.
- For all $A : R\text{-Alg}_{fp}$ the map $A \to R^{\mathsf{Spec}(A)}$ is an equivalence.
- For all $A : R\text{-Alg}_{fp}$ the set $\mathsf{Spec}(A)$ has choice.

Zariski topology

Definition

The Zariski topology consists of the types merely of the form

$$x_1 \text{ inv} + \cdots + x_n \text{ inv}$$

where $(x_1, \cdots, x_n) = 1$ in R.

Remark

R is local if and only if for all $S \in \text{Zar}$ we have $\|S\|$.

The Zariski sheaf model

Proposition

We have a model of HoTT where:

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- For all $A : R\text{-Alg}_{fp}$ the map $A \to R^{\mathrm{Spec}(A)}$ is an equivalence.
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An interlude: sheafifying in a sheaf model

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The same story works for a topology internal to sheaves.

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Remark

In Zariski sheaves, affines are stable under finite sums.

Now we work in Zariski sheaves and replace finite sums of affines by affines in the definition of topology.

Étale topology

Definition

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Proposition

A type X is an étale sheaf if and only if the map

$$X \to X^{\|\mathrm{Spec}(R[X]/P)\|}$$

is an equivalence for all P monic unramifiable.

The étale sheaf model

Proposition

We have a model of HoTT where:

- There is a local ring R such that any monic unramifiable polynomial in R has a root.
- For all $A : R\text{-Alg}_{fp}$ the map $A \to R^{\mathrm{Spec}(A)}$ is an equivalence.
- For all $A : R\text{-Alg}_{fp}$ the set $\mathrm{Spec}(A)$ has étale-local choice.

The fppf topology

Definition

A affine Spec(A) is flat if the functor $A \otimes_R \_$ is exact.

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Remark

Given P monic non-constant, Spec($R[X]/P$) is non-empty and flat.
So in fppf sheaves, monic non-constant polynomials have roots.

Remark

Formally étale affines are flat, so fppf sheaves are étale sheaves.

The fppf sheaf model

Proposition

We have a model of HoTT where:

- There is a local ring R such that any monic non-constant polynomial in R has a root.
- For all $A : R\text{-Alg}_{fp}$ the map $A \to R^{\mathrm{Spec}(A)}$ is an equivalence.
- For all $A : R\text{-Alg}_{fp}$ the set $\mathrm{Spec}(A)$ has flat-local choice.

The four models compared

Topology	The ring R is	Affines have
-	arbitrary	full choice
Zariski	local	Zariski-local choice
Étale	local + monic unramifiable have roots	étale-local choice
Fppf	local + monic non-constant have roots	flat-local choice

Outline

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Since boolean algebras are equivalent to boolean rings,
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Remark

The story we told with finitely presented algebras works similarly with countably presented algebras.

Since boolean algebras are equivalent to boolean rings, we can reuse the Zariski, étale and fppf topology as is.

Simplifications

- 2 is the only local boolean ring.
  So in Zariski sheaves $\mathbb{G} = 2$.
- Any boolean ring is formally étale and flat.
  So fppf topology = étale topology = $\{S \mid \neg\neg S\}$.

A model for synthetic Stone duality

Theorem

We have a model of HoTT where:

- For all $A : \mathsf{Bool}_{cp}$, if $\neg\neg\mathsf{Spec}(A)$ then $\|\mathsf{Spec}(A)\|$.
- For all $A : \mathsf{Bool}_{cp}$, the map $A \to 2^{\mathsf{Spec}(A)}$ is an equivalence.
- For all $A : \mathsf{Bool}_{cp}$, the set $\mathsf{Spec}(A)$ has local choice.

Triangulated type theory[14]

Let $\mathbb{T}$ be the theory of distributive lattice.
Let $\mathbb{L}$ be the generic such lattice.

Definition

Consider the topology generated by:

- If $0 =_{\mathbb{L}} 1$ we add $\bot$.
- For all $i, j : \mathbb{L}$ we add $i \leq j + j \leq i$.

[14] [Gratzer, Weinberger, Buchholtz 24]

Triangulated type theory[14]

Let $\mathbb{T}$ be the theory of distributive lattice.
Let $\mathbb{L}$ be the generic such lattice.

Definition

Consider the topology generated by:

- If $0 =_{\mathbb{L}} 1$ we add $\bot$.
- For all $i, j : \mathbb{L}$ we add $i \leq j + j \leq i$.

We should get a model of HoTT with:

- $\mathbb{L}$ a totally ordered bounded set.
- Duality for finitely presented distributive lattice over $\mathbb{L}$.
- Some form of local choice for affines.

[14] [Gratzer, Weinberger, Buchholtz 24]

Summary

Given spaces of interests build by gluing basic spaces such that:

- These basic spaces enjoy a duality with $\mathbb{T}$-algebras.
- The topology $\mathbb{J}$ encoding the gluing can be defined internally.
- $\mathbb{J}$ is subcanonical and simply defined.

Then we have a model of HoTT where:

- We have a $\mathbb{T}$-algebra $\mathbb{G}$.
- $\mathbb{J}$-covers are surjective.
- We have a duality between f.p. $\mathbb{G}$-algebras and affines.
- Affines have $\mathbb{J}$-local choice.

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In SAG and SSD, these axioms are enough for substantial results.