Pushforwards in Inverse Homotopical Diagrams

Krzysztof Kapulkin and Yufeng Li

Motivation

Background

Krzysztof Kapulkin, and Peter LeFanu Lumsdaine. The homotopy theory of type theories. Advances in Mathematics, 2018.

Morita equivalence

The category of models of ITT admits a cofibrantly-generated model structure

Motivation

Background

Krzysztof Kapulkin, and Peter LeFanu Lumsdaine. The homotopy theory of type theories. Advances in Mathematics, 2018.

Morita equivalence

The category of models of ITT admits a cofibrantly-generated model structure

- Cofibrations freely add terms and types

Motivation

Background

Krzysztof Kapulkin, and Peter LeFanu Lumsdaine. The homotopy theory of type theories. Advances in Mathematics, 2018.

Morita equivalence

The category of models of ITT admits a cofibrantly-generated model structure

- Cofibrations freely add terms and types
- Trivial cofibrations freely add propositional equalities and equivalences

Motivation

Background

Krzysztof Kapulkin, and Peter LeFanu Lumsdaine. The homotopy theory of type theories. Advances in Mathematics, 2018.

Morita equivalence

The category of models of ITT admits a cofibrantly-generated model structure

- Cofibrations freely add terms and types
- Trivial cofibrations freely add propositional equalities and equivalences
- Weak equivalences say that expressible and provable statements in those two models are correspond propositionally within type theory

Motivation

Background

Krzysztof Kapulkin, and Peter LeFanu Lumsdaine. The homotopy theory of type theories. Advances in Mathematics, 2018.

Morita equivalence

The category of models of ITT admits a cofibrantly-generated model structure

- Cofibrations freely add terms and types
- Trivial cofibrations freely add propositional equalities and equivalences
- Weak equivalences say that expressible and provable statements in those two models are correspond propositionally within type theory

$$\begin{pmatrix}\text{Equivalence of} \\ \text{type theories}\end{pmatrix} \overset{\text{def}}{=} \text{Morita equivalence} \overset{\text{def}}{=} \begin{pmatrix}\text{Quillen equivalence of} \\ \text{categories of models}\end{pmatrix}$$

Motivation

Background

Krzysztof Kapulkin, and Peter LeFanu Lumsdaine. The homotopy theory of type theories. Advances in Mathematics, 2018.

Krzysztof Kapulkin, and Y.L. Extensional concepts in intensional type theory, revisited. Theoretical Computer Science, 2025.

Morita equivalence

The category of models of ITT admits a cofibrantly-generated model structure

- Cofibrations freely add terms and types
- Trivial cofibrations freely add propositional equalities and equivalences
- Weak equivalences say that expressible and provable statements in those two models are correspond propositionally within type theory

$$\begin{pmatrix}\text{Equivalence of}\\ \text{type theories}\end{pmatrix} \stackrel{\text{def}}{=} \text{Morita equivalence} \stackrel{\text{def}}{=} \begin{pmatrix}\text{Quillen equivalence of}\\ \text{categories of models}\end{pmatrix}$$

Motivation

Background

Krzysztof Kapulkin, and Peter LeFanu Lumsdaine. The homotopy theory of type theories. Advances in Mathematics, 2018.

Krzysztof Kapulkin, and Y.L. Extensional concepts in intensional type theory, revisited. Theoretical Computer Science, 2025.

Morita equivalence

The category of models of ITT admits a cofibrantly-generated model structure

- Cofibrations freely add terms and types
- Trivial cofibrations freely add propositional equalities and equivalences
- Weak equivalences say that expressible and provable statements in those two models are correspond propositionally within type theory

$$\left(\begin{array}{c}\text{Equivalence of}\\ \text{type theories}\end{array}\right) \overset{\text{def}}{=} \text{Morita equivalence} \overset{\text{def}}{=} \left(\begin{array}{c}\text{Quillen equivalence of}\\ \text{categories of models}\end{array}\right)$$

Path object construction

- The path object model for each $\mathbb{C} \in \mathrm{Mod}_{\mathrm{ITT}}$ is an inverse homotopical diagram category with Reedy fibrations

Motivation

Background

Krzysztof Kapulkin, and Peter LeFanu Lumsdaine. The homotopy theory of type theories. Advances in Mathematics, 2018.

Krzysztof Kapulkin, and Y.L. Extensional concepts in intensional type theory, revisited. Theoretical Computer Science, 2025.

Morita equivalence

The category of models of ITT admits a cofibrantly-generated model structure

- Cofibrations freely add terms and types
- Trivial cofibrations freely add propositional equalities and equivalences
- Weak equivalences say that expressible and provable statements in those two models are correspond propositionally within type theory

$$\left(\begin{array}{c}\text{Equivalence of}\\ \text{type theories}\end{array}\right) \overset{\text{def}}{=} \text{Morita equivalence} \overset{\text{def}}{=} \left(\begin{array}{c}\text{Quillen equivalence of}\\ \text{categories of models}\end{array}\right)$$

Path object construction

- The path object model for each $\mathbb{C} \in \mathsf{Mod}_{\mathsf{ITT}}$ is an inverse homotopical diagram category with Reedy fibrations
- Must ensure the path object model supports various logical structures: most challenging of which is the Π-types (i.e. pushforwards)

Background and references

Background

Krzysztof Kapulkin, and Peter LeFanu Lumsdaine. Homotopical inverse diagrams in categories with attributes. Journal of Pure and Applied Algebra, 2021.

Marcelo Fiore, Krzysztof Kapulkin, and Y.L. Logical structures in inverse functor categories. arxiv:2410.11728, 2024.

Main result

Krzysztof Kapulkin and Y.L. Pushforwards in Inverse Homotopical Diagrams. arxiv:2506.04472, 2025.

Definition

An inverse category $\mathcal{I}$ is a category with a grading $\deg\colon \mathcal{I} \to \mathbb{N}$ such that non-identities strictly decrease degree:

Definition

An inverse category $\mathcal{I}$ is a category with a grading $\mathsf{deg}\colon \mathcal{I} \to \mathbb{N}$ such that non-identities strictly decrease degree:

$$(i \longrightarrow j) \in \mathcal{I}$$

Inverse diagram categories

Definition

An inverse category $\mathcal{I}$ is a category with a grading $\deg\colon \mathcal{I} \to \mathbb{N}$ such that non-identities strictly decrease degree:

$$\mathrm{id} \neq (i \longrightarrow j) \in \mathcal{I}$$

Inverse diagram categories

Definition

An inverse category $\mathcal{I}$ is a category with a grading $\deg\colon \mathcal{I} \to \mathbb{N}$ such that non-identities strictly decrease degree:

$$\mathrm{id} \neq (i \longrightarrow j) \in \mathcal{I} \Longrightarrow \deg(i) > \deg(j)$$

Inverse diagram categories

Definition

An inverse category $\mathcal{I}$ is a category with a grading $\mathsf{deg}\colon \mathcal{I} \to \mathbb{N}$ such that non-identities strictly decrease degree:

$$\mathrm{id} \neq (i \longrightarrow j) \in \mathcal{I} \Longrightarrow \mathsf{deg}(i) > \mathsf{deg}(j)$$

Example. The walking span $\{0 \leftarrow 01 \rightarrow 1\}$ is an inverse category.

Inverse diagram categories

Definition

An inverse category $\mathcal{I}$ is a category with a grading $\deg\colon \mathcal{I} \to \mathbb{N}$ such that non-identities strictly decrease degree:

$$\mathrm{id} \neq (i \longrightarrow j) \in \mathcal{I} \Longrightarrow \deg(i) > \deg(j)$$

Example. The walking span $\{0 \leftarrow 01 \rightarrow 1\}$ is an inverse category.

New models from old

If $\mathbb{C}$ is a model of logic then one would like $\mathbb{C}^{\mathcal{I}}$ to be a model of logic, for inverse categories $\mathcal{I}$.

Inverse diagram categories

Definition

An inverse category $\mathcal{I}$ is a category with a grading $\deg\colon \mathcal{I} \to \mathbb{N}$ such that non-identities strictly decrease degree:

$$\mathrm{id} \neq (i \longrightarrow j) \in \mathcal{I} \Longrightarrow \deg(i) > \deg(j)$$

Example. The walking span $\{0 \leftarrow 01 \rightarrow 1\}$ is an inverse category.

New models from old

If $\mathbb{C}$ is a model of logic then one would like $\mathbb{C}^{\mathcal{I}}$ to be a model of logic, for inverse categories $\mathcal{I}$.

- $\mathbb{C}^{\mathcal{I}}$ can often validate or refute additional axioms: $\mathsf{Set}^{\rightarrow}$ refutes excluded middle

Inverse diagram categories

Definition

An inverse category $\mathcal{I}$ is a category with a grading $\deg\colon \mathcal{I} \to \mathbb{N}$ such that non-identities strictly decrease degree:

$$\mathrm{id} \neq (i \longrightarrow j) \in \mathcal{I} \implies \deg(i) > \deg(j)$$

Example. The walking span $\{0 \leftarrow 01 \rightarrow 1\}$ is an inverse category.

New models from old

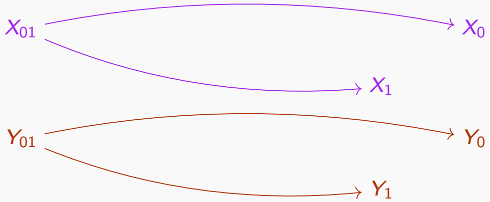
If $\mathbb{C}$ is a model of logic then one would like $\mathbb{C}^{\mathcal{I}}$ to be a model of logic, for inverse categories $\mathcal{I}$.

- $\mathbb{C}^{\mathcal{I}}$ can often validate or refute additional axioms: $\mathsf{Set}^{\rightarrow}$ refutes excluded middle
- The path object for each $\mathbb{C} \in \mathsf{Mod}_{\mathsf{ITT}}$ is an inverse homotopical diagram category with Reedy fibrations in the Kapulkin–Lumsdaine model structure

Reedy diagrams inductively

Constructing spans inductively

Suppose we want to construct $k\colon X \to Y$ in $\mathbb{C}^{0\leftarrow 01 \rightarrow 1}$



Reedy diagrams inductively

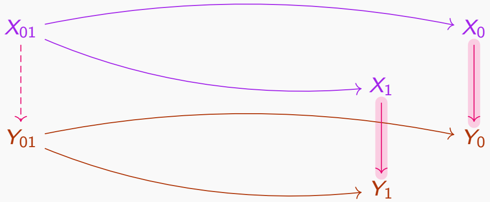
Constructing spans inductively

Suppose we want to construct $k\colon X \to Y$ in $\mathbb{C}^{0 \leftarrow 01 \rightarrow 1}$

Reedy diagrams inductively

Constructing spans inductively

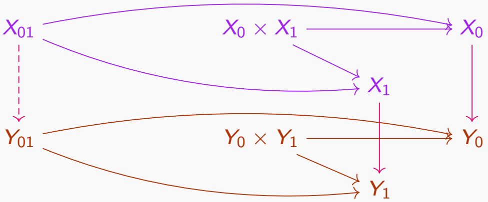
Suppose we want to construct $k\colon X \to Y$ in $\mathbb{C}^{0\leftarrow 01 \rightarrow 1}$ from components at $0, 1$.



Reedy diagrams inductively

Constructing spans inductively

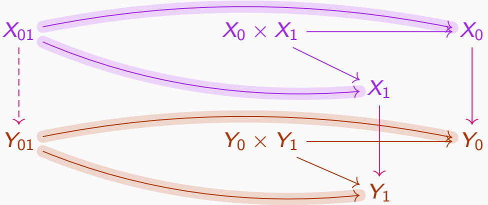
Suppose we want to construct $k\colon X \to Y$ in $\mathbb{C}^{0 \leftarrow 01 \rightarrow 1}$ from components at $0, 1$.



Reedy diagrams inductively

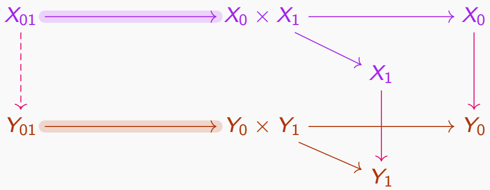
Constructing spans inductively

Suppose we want to construct $k\colon X \to Y$ in $\mathbb{C}^{0\leftarrow 01\rightarrow 1}$ from components at $0, 1$.



Constructing spans inductively

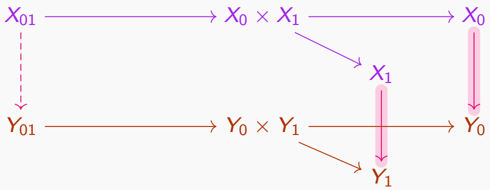
Suppose we want to construct $k\colon X \to Y$ in $\mathbb{C}^{0\leftarrow 01 \rightarrow 1}$ from components at $0, 1$.



Reedy diagrams inductively

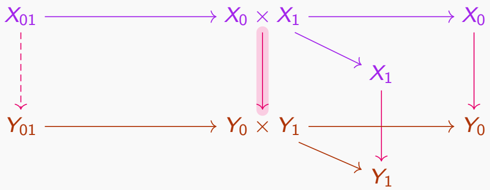
Constructing spans inductively

Suppose we want to construct $k\colon X \to Y$ in $\mathbb{C}^{0 \leftarrow 01 \rightarrow 1}$ from components at $0, 1$.



Constructing spans inductively

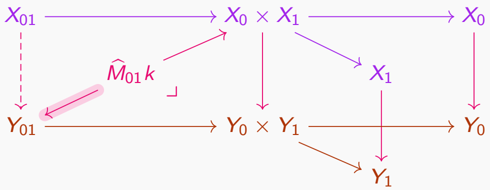
Suppose we want to construct $k\colon X \to Y$ in $\mathbb{C}^{0 \leftarrow 01 \rightarrow 1}$ from components at $0, 1$.



Reedy diagrams inductively

Constructing spans inductively

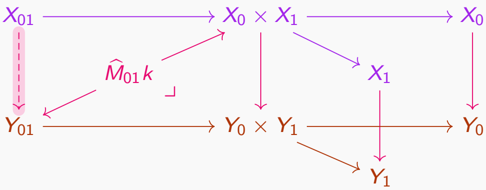
Suppose we want to construct $k\colon X \to Y$ in $\mathbb{C}^{0\leftarrow 01 \rightarrow 1}$ from components at $0, 1$.



Reedy diagrams inductively

Constructing spans inductively

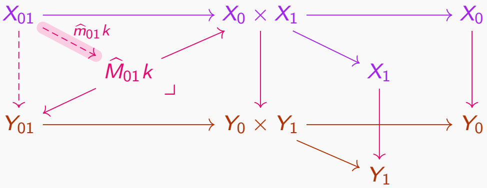
Suppose we want to construct $k\colon X \to Y$ in $\mathbb{C}^{0\leftarrow 01\rightarrow 1}$ from components at $0, 1$. Then the component at 01



Reedy diagrams inductively

Constructing spans inductively

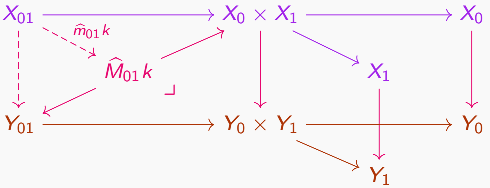
Suppose we want to construct $k\colon X \to Y$ in $\mathbb{C}^{0 \leftarrow 01 \rightarrow 1}$ from components at $0, 1$.
Then the component at 01 is uniquely a map $\widehat{m}_{01} k\colon X_{01} \to \widehat{M}_{01} k$.



Reedy diagrams inductively

Constructing spans inductively

Suppose we want to construct $k\colon X \to Y$ in $\mathbb{C}^{0\leftarrow 01 \rightarrow 1}$ from components at $0, 1$.
Then the component at 01 is uniquely a map $\widehat{m}_{01}k\colon X_{01} \to \widehat{M}_{01}k$.



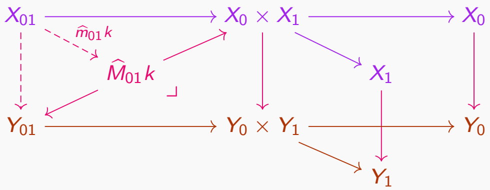
Constructing diagrams inductively

Generally, for inverse $\mathcal{I}$, constructing the i-th component of construct $k\colon X \to Y$

Reedy diagrams inductively

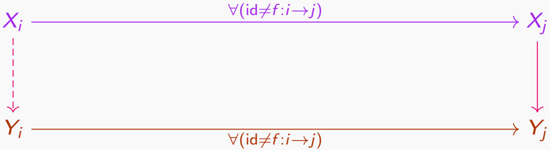
Constructing spans inductively

Suppose we want to construct $k\colon X \to Y$ in $\mathbb{C}^{0\leftarrow 01 \rightarrow 1}$ from components at $0, 1$. Then the component at 01 is uniquely a map $\widehat{m}_{01} k\colon X_{01} \to \widehat{M}_{01} k$.



Constructing diagrams inductively

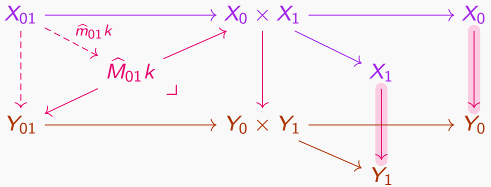
Generally, for inverse $\mathcal{I}$, constructing the i-th component of construct $k\colon X \to Y$ from those at lower degrees j



Reedy diagrams inductively

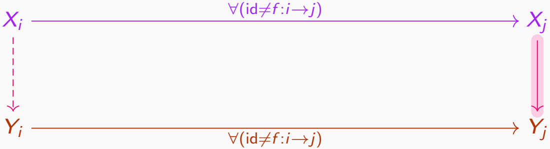
Constructing spans inductively

Suppose we want to construct $k\colon X \to Y$ in $\mathbb{C}^{0 \leftarrow 01 \rightarrow 1}$ from components at $0, 1$. Then the component at 01 is uniquely a map $\widehat{m}_{01}k\colon X_{01} \to \widehat{M}_{01}k$.



Constructing diagrams inductively

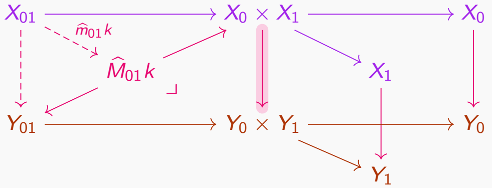
Generally, for inverse $\mathcal{I}$, constructing the i-th component of construct $k\colon X \to Y$ from those at lower degrees j



Reedy diagrams inductively

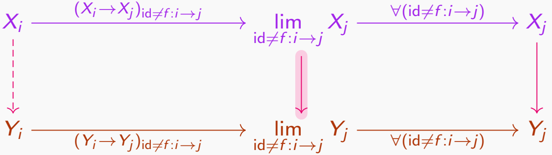
Constructing spans inductively

Suppose we want to construct $k\colon X \to Y$ in $\mathbb{C}^{0\leftarrow 01 \rightarrow 1}$ from components at $0, 1$.
Then the component at 01 is uniquely a map $\widehat{m}_{01}k\colon X_{01} \to \widehat{M}_{01}k$.



Constructing diagrams inductively

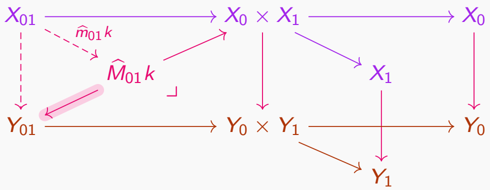
Generally, for inverse $\mathcal{I}$, constructing the i-th component of construct $k\colon X \to Y$ from those at lower degrees j



Reedy diagrams inductively

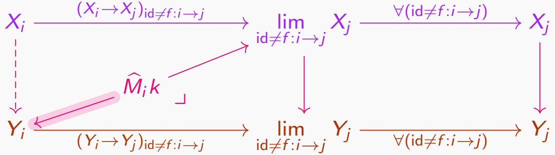
Constructing spans inductively

Suppose we want to construct $k\colon X \to Y$ in $\mathbb{C}^{0\leftarrow 01 \rightarrow 1}$ from components at $0, 1$. Then the component at 01 is uniquely a map $\widehat{m}_{01}k\colon X_{01} \to \widehat{M}_{01}k$.



Constructing diagrams inductively

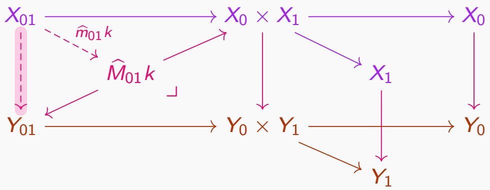
Generally, for inverse $\mathcal{I}$, constructing the i-th component of construct $k\colon X \to Y$ from those at lower degrees j



Reedy diagrams inductively

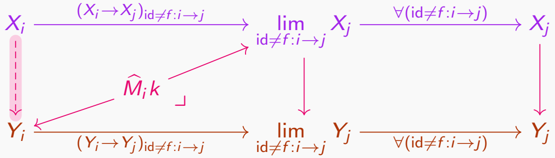
Constructing spans inductively

Suppose we want to construct $k\colon X \to Y$ in $\mathbb{C}^{0\leftarrow 01 \rightarrow 1}$ from components at $0, 1$. Then the component at 01 is uniquely a map $\widehat{m}_{01} k\colon X_{01} \to \widehat{M}_{01} k$.



Constructing diagrams inductively

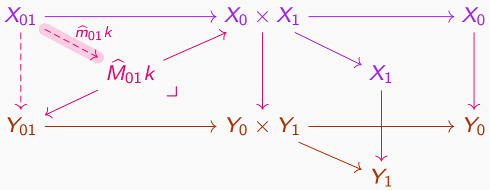
Generally, for inverse $\mathcal{I}$, constructing the i-th component of construct $k\colon X \to Y$ from those at lower degrees j



Reedy diagrams inductively

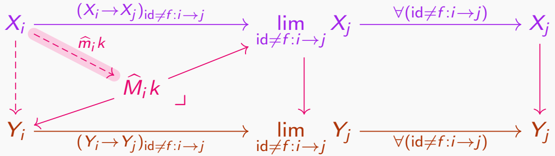
Constructing spans inductively

Suppose we want to construct $k\colon X \to Y$ in $\mathbb{C}^{0\leftarrow 01\rightarrow 1}$ from components at $0, 1$. Then the component at 01 is uniquely a map $\widehat{m}_{01}k\colon X_{01} \to \widehat{M}_{01}k$.

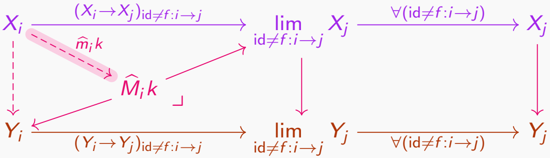


Constructing diagrams inductively

Generally, for inverse $\mathcal{I}$, constructing the i-th component of construct $k\colon X \to Y$ from those at lower degrees j amounts to a map $\widehat{m}_i k\colon X_i \to \widehat{M}_i k$

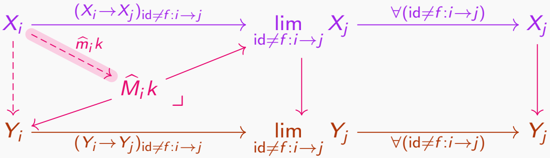


Definition


X_i
$(X_i \to X_j)_{\mathrm{id} \neq f : i \to j}$
$\lim_{\mathrm{id} \neq f : i \to j} X_j$
$\forall(\mathrm{id} \neq f : i \to j)$
X_j
$\widehat{m}_i k$
$\widehat{M}_i k$
Y_i
$(Y_i \to Y_j)_{\mathrm{id} \neq f : i \to j}$
$\lim_{\mathrm{id} \neq f : i \to j} Y_j$
$\forall(\mathrm{id} \neq f : i \to j)$
Y_j

Definition

Suppose $\mathbb{C}$ is equipped with a class of pullback-stable fibrations.

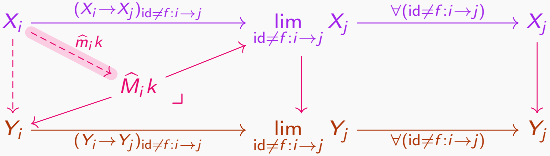


Reedy fibrations and homotopical diagrams

Definition

Suppose $\mathbb{C}$ is equipped with a class of pullback-stable fibrations.

For inverse $\mathcal{I}$, a map $k\colon X \to Y$ in $\mathbb{C}^{\mathcal{I}}$ is a Reedy fibration when each $\widehat{m}_i k\colon X_i \to \widehat{M}_i k$ is a fibration in $\mathbb{C}$.

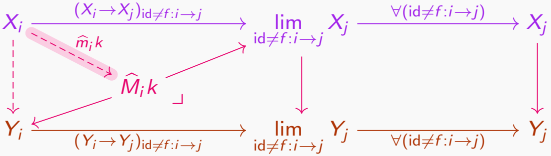


Reedy fibrations and homotopical diagrams

Definition

Suppose $\mathbb{C}$ is equipped with a class of pullback-stable fibrations.

For inverse $\mathcal{I}$, a map $k\colon X \to Y$ in $\mathbb{C}^{\mathcal{I}}$ is a Reedy fibration when each $\widehat{m}_i k\colon X_i \to \widehat{M}_i k$ is a fibration in $\mathbb{C}$.



Definition

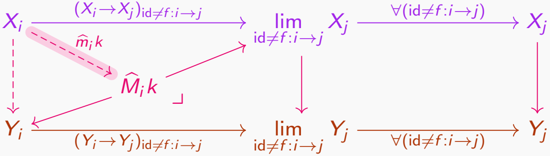
A category with weak equivalences is a category $\mathbb{C}$ equipped with a replete wide subcategory $\mathcal{W}_{\mathbb{C}} \hookrightarrow \mathbb{C}$ closed under two-out-of-three called the weak equivalences.

Reedy fibrations and homotopical diagrams

Definition

Suppose $\mathbb{C}$ is equipped with a class of pullback-stable fibrations.

For inverse $\mathcal{I}$, a map $k\colon X \to Y$ in $\mathbb{C}^{\mathcal{I}}$ is a Reedy fibration when each $\widehat{m}_i k\colon X_i \to \widehat{M}_i k$ is a fibration in $\mathbb{C}$.



Definition

A category with weak equivalences is a category $\mathbb{C}$ equipped with a replete wide subcategory $\mathcal{W}_{\mathbb{C}} \hookrightarrow \mathbb{C}$ closed under two-out-of-three called the weak equivalences.

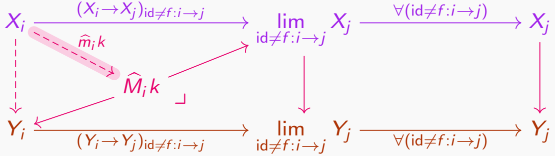
A homotopical diagram $(\mathcal{I}, \mathcal{W}_{\mathcal{I}}) \to (\mathbb{C}, \mathcal{W}_{\mathbb{C}})$ is a diagram $\mathcal{I} \to \mathbb{C}$ preserving weak equivalences.

Reedy fibrations and homotopical diagrams

Definition

Suppose $\mathbb{C}$ is equipped with a class of pullback-stable fibrations.

For inverse $\mathcal{I}$, a map $k \colon X \to Y$ in $\mathbb{C}^{\mathcal{I}}$ is a Reedy fibration when each $\widehat{m}_i k \colon X_i \to \widehat{M}_i k$ is a fibration in $\mathbb{C}$.



Definition

A category with weak equivalences is a category $\mathbb{C}$ equipped with a replete wide subcategory $\mathcal{W}_{\mathbb{C}} \hookrightarrow \mathbb{C}$ closed under two-out-of-three called the weak equivalences.

A homotopical diagram $(\mathcal{I}, \mathcal{W}_{\mathcal{I}}) \to (\mathbb{C}, \mathcal{W}_{\mathbb{C}})$ is a diagram $\mathcal{I} \to \mathbb{C}$ preserving weak equivalences.

Example. In the Kapulkin–Lumsdaine model structure on the category of models of ITT, the path object of each model $\mathbb{C}$ are the homotopical spans $\mathbb{C}^{0 \leftarrow 01 \rightarrow 1}$ with both legs weak equivalences.

Problem statement

Definition

For finitely complete $\mathbb{C}$, the pushforward along $q \colon B \to A \in \mathbb{C}$ is a right adjoint to the pullback along q.

Problem statement

Definition

For finitely complete $\mathbb{C}$, the pushforward along $q\colon B \to A \in \mathbb{C}$ is a right adjoint to the pullback along q.

Example. For $X, E \rightrightarrows B$, the local exponential $[X, E]_B$ is the pushforward of $X \times_B E \to E$ along $E \to B$.

Problem statement

Definition

For finitely complete $\mathbb{C}$, the pushforward along $q\colon B \to A \in \mathbb{C}$ is a right adjoint to the pullback along q.

Example. For $X, E \rightrightarrows B$, the local exponential $[X, E]_B$ is the pushforward of $X \times_B E \to E$ along $E \to B$.

Problem statement

Suppose $(\mathbb{C}, \mathcal{W}_{\mathbb{C}})$ is a category with weak equivalences equipped with a pullback-stable class of fibrations where pushforwards of fibrations along fibrations exists.

Problem statement

Definition

For finitely complete $\mathbb{C}$, the pushforward along $q\colon B \to A \in \mathbb{C}$ is a right adjoint to the pullback along q.

Example. For $X, E \rightrightarrows B$, the local exponential $[X, E]_B$ is the pushforward of $X \times_B E \to E$ along $E \to B$.

Problem statement

Suppose $(\mathbb{C}, \mathcal{W}_{\mathbb{C}})$ is a category with weak equivalences equipped with a pullback-stable class of fibrations where pushforwards of fibrations along fibrations exists.

Then for an inverse homotopical category $(\mathcal{I}, \mathcal{W}_{\mathcal{I}})$

Problem statement

Definition

For finitely complete $\mathbb{C}$, the pushforward along $q \colon B \to A \in \mathbb{C}$ is a right adjoint to the pullback along q.

Example. For $X, E \rightrightarrows B$, the local exponential $[X, E]_B$ is the pushforward of $X \times_B E \to E$ along $E \to B$.

Problem statement

Suppose $(\mathbb{C}, \mathcal{W}_{\mathbb{C}})$ is a category with weak equivalences equipped with a pullback-stable class of fibrations where pushforwards of fibrations along fibrations exists.

Then for an inverse homotopical category $(\mathcal{I}, \mathcal{W}_{\mathcal{I}})$ the category of homotopical diagrams $\mathcal{I} \to \mathbb{C}$

Problem statement

Definition

For finitely complete $\mathbb{C}$, the pushforward along $q\colon B \to A \in \mathbb{C}$ is a right adjoint to the pullback along q.

Example. For $X, E \rightrightarrows B$, the local exponential $[X, E]_B$ is the pushforward of $X \times_B E \to E$ along $E \to B$.

Problem statement

Suppose $(\mathbb{C}, \mathcal{W}_{\mathbb{C}})$ is a category with weak equivalences equipped with a pullback-stable class of fibrations where pushforwards of fibrations along fibrations exists.

Then for an inverse homotopical category $(\mathcal{I}, \mathcal{W}_{\mathcal{I}})$, when is the category of homotopical diagrams $\mathcal{I} \to \mathbb{C}$ closed under pushforwards of Reedy fibrations along Reedy fibrations?

Problem statement

Definition

For finitely complete $\mathbb{C}$, the pushforward along $q\colon B \to A \in \mathbb{C}$ is a right adjoint to the pullback along q.

Example. For $X, E \rightrightarrows B$, the local exponential $[X, E]_B$ is the pushforward of $X \times_B E \to E$ along $E \to B$.

Problem statement

Suppose $(\mathbb{C}, \mathcal{W}_{\mathbb{C}})$ is a category with weak equivalences equipped with a pullback-stable class of fibrations where pushforwards of fibrations along fibrations exists.

Then for an inverse homotopical category $(\mathcal{I}, \mathcal{W}_{\mathcal{I}})$, when is the category of homotopical diagrams $\mathcal{I} \to \mathbb{C}$ closed under pushforwards of Reedy fibrations along Reedy fibrations?

Theorem (Shulman 2015)

Homotopical diagrams $\mathcal{I} \to \mathbb{C}$ closed under pushforwards of Reedy fibrations along Reedy fibrations when no maps of $\mathcal{I}$ are weak equivalences.

Problem statement

Definition

For finitely complete $\mathbb{C}$, the pushforward along $q \colon B \to A \in \mathbb{C}$ is a right adjoint to the pullback along q.

Example. For $X, E \rightrightarrows B$, the local exponential $[X, E]_B$ is the pushforward of $X \times_B E \to E$ along $E \to B$.

Problem statement

Suppose $(\mathbb{C}, \mathcal{W}_{\mathbb{C}})$ is a category with weak equivalences equipped with a pullback-stable class of fibrations where pushforwards of fibrations along fibrations exists.

Then for an inverse homotopical category $(\mathcal{I}, \mathcal{W}_{\mathcal{I}})$, when is the category of homotopical diagrams $\mathcal{I} \to \mathbb{C}$ closed under pushforwards of Reedy fibrations along Reedy fibrations?

Theorem (Shulman 2015)

Homotopical diagrams $\mathcal{I} \to \mathbb{C}$ closed under pushforwards of Reedy fibrations along Reedy fibrations when no maps of $\mathcal{I}$ are weak equivalences.

Theorem (Kapulkin–Lumsdaine 2021)

Homotopical diagrams $\mathcal{I} \to \mathbb{C}$ closed under pushforwards of Reedy fibrations along Reedy fibrations when all maps of $\mathcal{I}$ are weak equivalences.

Exponentials of spans

Exponentials of spans

$$X_0 \longleftarrow X_{01} \longrightarrow X_1$$

$$Y_0 \longleftarrow Y_{01} \longrightarrow Y_1$$

Exponentials of spans

Exponentials of spans

$$X_0 \longleftarrow X_{01} \longrightarrow X_1$$

$$Y_0 \longleftarrow Y_{01} \longrightarrow Y_1$$

$$[X, Y]_0 \dashleftarrow [X, Y]_{01} \dashrightarrow [X, Y]_1$$

Exponentials of spans

Exponentials of spans

$$X_0 \longleftarrow X_{01} \longrightarrow X_1$$

$$Y_0 \longleftarrow Y_{01} \longrightarrow Y_1$$

$$[X, Y]_{01}$$

$$\downarrow$$

$$[X, Y]_0 \times [X, Y]_1$$

Exponentials of spans

$$\begin{array}{ccccc} X_0 & \longleftarrow & X_{01} & \longrightarrow & X_1 \\ \downarrow & & \downarrow & & \downarrow \\ Y_0 & \longleftarrow & Y_{01} & \longrightarrow & Y_1 \end{array}$$

$$\begin{array}{c} [X, Y]_{01} \\ \downarrow \\ [X, Y]_0 \times [X, Y]_1 \end{array}$$

Exponentials of spans

Exponentials of spans

$$\begin{array}{ccccc} X_0 & \longleftarrow & X_{01} & \longrightarrow & X_1 \\ \downarrow & & \downarrow & & \downarrow \\ Y_0 & \longleftarrow & Y_{01} & \longrightarrow & Y_1 \end{array}$$

$$\begin{array}{c} [X, Y]_{01} \\ \downarrow \\ [X, Y]_0 \times [X, Y]_1 \end{array}$$

Exponentials of spans

Exponentials of spans

$$\begin{array}{ccccc} X_0 & \longleftarrow & X_{01} & \longrightarrow & X_1 \\ \downarrow & & \downarrow & & \downarrow \\ Y_0 & \longleftarrow & Y_{01} & \longrightarrow & Y_1 \end{array}$$

$$\begin{array}{l} [X, Y]_{01} \\ \quad \downarrow \\ [X, Y]_0 \times [X, Y]_1 \stackrel{=}{\Longrightarrow} [X_0, Y_0] \times [X_1, Y_1] \end{array}$$

Exponentials of spans

$$
\begin{array}{ccccc}
X_0 & \longleftarrow & X_{01} & \longrightarrow & X_1 \\
\downarrow & & \downarrow & & \downarrow \\
Y_0 & \longleftarrow & Y_{01} & \longrightarrow & Y_1
\end{array}
$$

$$
\begin{array}{l}
[X, Y]_{01} \\
\downarrow \\
[X, Y]_0 \times [X, Y]_1 \Rightarrow [X_0, Y_0] \times [X_1, Y_1]
\end{array}
$$

Exponentials of spans

Exponentials of spans

$$\begin{array}{ccccc} X_0 & \longleftarrow & X_{01} & \longrightarrow & X_1 \\ \downarrow & & \downarrow & & \downarrow \\ Y_0 & \longleftarrow & Y_{01} & \longrightarrow & Y_1 \end{array}$$

$$\begin{array}{ccc} [X, Y]_{01} & \dashrightarrow & [X_{01}, Y_{01}] \\ \downarrow & & \\ [X, Y]_0 \times [X, Y]_1 & \overset{=}{\Longrightarrow} & [X_0, Y_0] \times [X_1, Y_1] \end{array}$$

Exponentials of spans

$$\begin{array}{ccccc} X_0 & \leftarrow & X_{01} & \rightarrow & X_1 \\ \downarrow & & \downarrow & & \downarrow \\ Y_0 & \leftarrow & Y_{01} & \rightarrow & Y_1 \end{array}$$

$$\begin{array}{ccc} [X, Y]_{01} & \longrightarrow & [X_{01}, Y_{01}] \\ \downarrow & & \downarrow \\ [X, Y]_0 \times [X, Y]_1 \Rightarrow [X_0, Y_0] \times [X_1, Y_1] & & [X_{01}, Y_0] \times [X_{01}, Y_1] \end{array}$$

Exponentials of spans

$$
\begin{array}{ccccc}
X_0 & \longleftarrow & X_{01} & \longrightarrow & X_1 \\
\downarrow & & \downarrow & & \downarrow \\
Y_0 & \longleftarrow & Y_{01} & \longrightarrow & Y_1
\end{array}
$$

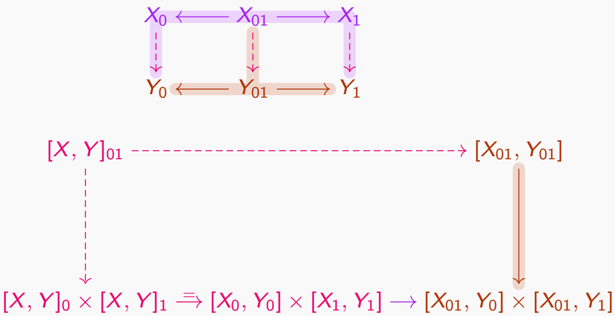
$$
\begin{array}{ccc}
[X, Y]_{01} & \dashrightarrow & [X_{01}, Y_{01}] \\
\downarrow & & \downarrow \\
[X, Y]_0 \times [X, Y]_1 \overset{=}{\Longrightarrow} [X_0, Y_0] \times [X_1, Y_1] & & [X_{01}, Y_0] \times [X_{01}, Y_1]
\end{array}
$$

Exponentials of spans

$$\begin{array}{ccccc} X_0 & \longleftarrow & X_{01} & \longrightarrow & X_1 \\ \downarrow & & \downarrow & & \downarrow \\ Y_0 & \longleftarrow & Y_{01} & \longrightarrow & Y_1 \end{array}$$

$$\begin{array}{ccccc} [X, Y]_{01} & & \dashrightarrow & & [X_{01}, Y_{01}] \\ \downarrow & & & & \downarrow \\ [X, Y]_0 \times [X, Y]_1 & \Rightarrow & [X_0, Y_0] \times [X_1, Y_1] & \longrightarrow & [X_{01}, Y_0] \times [X_{01}, Y_1] \end{array}$$

Exponentials of spans



Exponentials of spans

$$\begin{array}{ccccc} X_0 & \longleftarrow & X_{01} & \longrightarrow & X_1 \\ \downarrow & & \downarrow & & \downarrow \\ Y_0 & \longleftarrow & Y_{01} & \longrightarrow & Y_1 \end{array}$$

$$\begin{array}{ccccc} [X,Y]_{01} & & \dashrightarrow & & [X_{01},Y_{01}] \\ \downarrow & & & & \downarrow \\ [X,Y]_0 \times [X,Y]_1 & \xrightarrow{=} & [X_0,Y_0] \times [X_1,Y_1] & \longrightarrow & [X_{01},Y_0] \times [X_{01},Y_1] \end{array}$$

Exponentials of spans

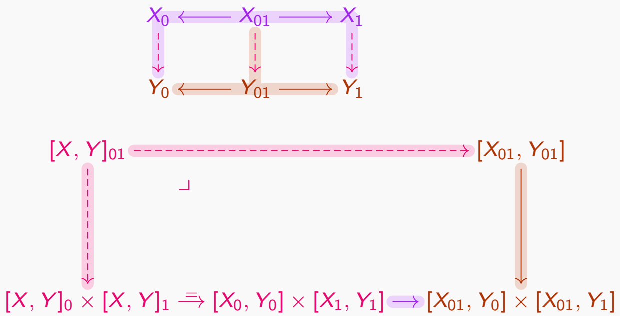
Exponentials of spans

$$
\begin{array}{ccccc}
X_0 & \longleftarrow & X_{01} & \longrightarrow & X_1 \\
\downarrow & & \downarrow & & \downarrow \\
Y_0 & \longleftarrow & Y_{01} & \longrightarrow & Y_1
\end{array}
$$

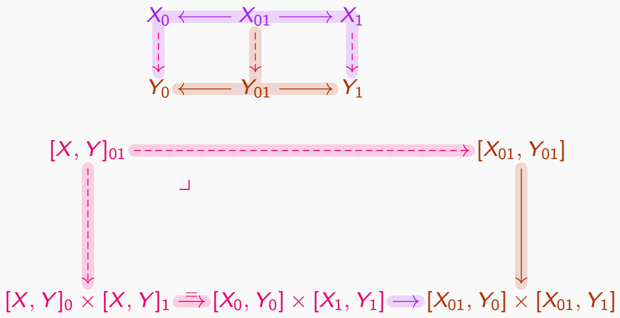
$$
\begin{array}{ccccc}
[X, Y]_{01} & & \longrightarrow & & [X_{01}, Y_{01}] \\
\downarrow & & & & \downarrow \\
[X, Y]_0 \times [X, Y]_1 & \Longrightarrow & [X_0, Y_0] \times [X_1, Y_1] & \longrightarrow & [X_{01}, Y_0] \times [X_{01}, Y_1]
\end{array}
$$

Exponentials of spans

Exponentials of spans



Exponentials of spans



Inductively constructing diagram exponentials

Proposition

Let $(\mathcal{I}, \mathsf{deg})$ be an inverse category and $\mathbb{C}$ be a cartesian closed category. Then, for diagrams $X, Y \in \mathbb{C}^{\mathcal{I}}$, the exponential diagram $[X, Y]$ is constructed by way of induction

Proposition

Let $(\mathcal{I}, \deg)$ be an inverse category and $\mathbb{C}$ be a cartesian closed category. Then, for diagrams $X, Y \in \mathbb{C}^{\mathcal{I}}$, the exponential diagram $[X, Y]$ is constructed by way of induction so that at each $i \in \mathcal{I}$, one has a comparison map

$$[X, Y]_i \xrightarrow{\kappa_i} [X_i, Y_i]$$

Inductively constructing diagram exponentials

Proposition

Let $(\mathcal{I}, \deg)$ be an inverse category and $\mathbb{C}$ be a cartesian closed category. Then, for diagrams $X, Y \in \mathbb{C}^{\mathcal{I}}$, the exponential diagram $[X, Y]$ is constructed by way of induction so that at each $i \in \mathcal{I}$, one has a comparison map

$$[X, Y]_i \xrightarrow{\kappa_i} [X_i, Y_i]$$

- For $\deg(i) = 0$, one puts $[X, Y]_0 \coloneqq [X_0, Y_0]$

Inductively constructing diagram exponentials

Proposition

Let $(\mathcal{I}, \deg)$ be an inverse category and $\mathbb{C}$ be a cartesian closed category. Then, for diagrams $X, Y \in \mathbb{C}^{\mathcal{I}}$, the exponential diagram $[X, Y]$ is constructed by way of induction so that at each $i \in \mathcal{I}$, one has a comparison map

$$[X, Y]_i \xrightarrow{\kappa_i} [X_i, Y_i]$$

- For $\deg(i) = 0$, one puts $[X, Y]_0 \coloneqq [X_0, Y_0] \xrightarrow{\kappa_i \coloneqq \mathrm{id}} [X_0, Y_0]$

Inductively constructing diagram exponentials

Proposition

Let $(\mathcal{I}, \deg)$ be an inverse category and $\mathbb{C}$ be a cartesian closed category. Then, for diagrams $X, Y \in \mathbb{C}^{\mathcal{I}}$, the exponential diagram $[X, Y]$ is constructed by way of induction so that at each $i \in \mathcal{I}$, one has a comparison map

$$[X, Y]_i \xrightarrow{\kappa_i} [X_i, Y_i]$$

- For $\deg(i) = 0$, one puts $[X, Y]_0 \coloneqq [X_0, Y_0] \xrightarrow{\kappa_i \coloneqq \mathrm{id}} [X_0, Y_0]$
- For $\deg(i) = n + 1$, inductively get $[X, Y]$ defined on $\partial({}^i/\mathcal{I}) = {}^i/\mathcal{I} - \{\mathrm{id}_i\}$ and $\kappa_j \colon [X, Y]_j \longrightarrow [X_j, Y_j]$ for $\mathrm{id} \neq f \colon i \to j \in \partial({}^i/\mathcal{I})$

$$[X, Y]_j \xrightarrow[\kappa_j]{} [X_j, Y_j]$$

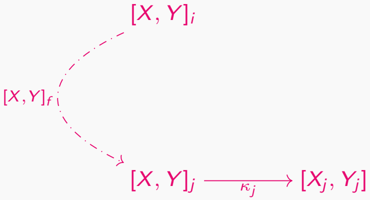
Inductively constructing diagram exponentials

Proposition

Let $(\mathcal{I}, \deg)$ be an inverse category and $\mathbb{C}$ be a cartesian closed category. Then, for diagrams $X, Y \in \mathbb{C}^{\mathcal{I}}$, the exponential diagram $[X, Y]$ is constructed by way of induction so that at each $i \in \mathcal{I}$, one has a comparison map

$$[X, Y]_i \xrightarrow{\kappa_i} [X_i, Y_i]$$

- For $\deg(i) = 0$, one puts $[X, Y]_0 := [X_0, Y_0] \xrightarrow{\kappa_i := \mathrm{id}} [X_0, Y_0]$
- For $\deg(i) = n + 1$, inductively get $[X, Y]$ defined on $\partial({}^{i}/\mathcal{I}) = {}^{i}/\mathcal{I} - \{\mathrm{id}_i\}$ and $\kappa_j \colon [X, Y]_j \longrightarrow [X_j, Y_j]$ for $\mathrm{id} \neq f \colon i \to j \in \partial({}^{i}/\mathcal{I})$



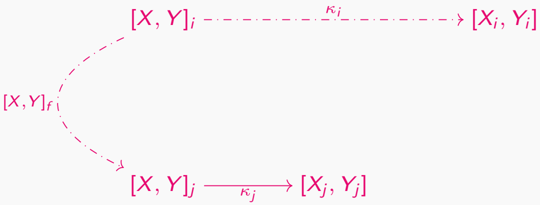
Inductively constructing diagram exponentials

Proposition

Let $(\mathcal{I}, \mathrm{deg})$ be an inverse category and $\mathbb{C}$ be a cartesian closed category. Then, for diagrams $X, Y \in \mathbb{C}^{\mathcal{I}}$, the exponential diagram $[X, Y]$ is constructed by way of induction so that at each $i \in \mathcal{I}$, one has a comparison map

$$[X, Y]_i \xrightarrow{\kappa_i} [X_i, Y_i]$$

- For $\mathrm{deg}(i) = 0$, one puts $[X, Y]_0 := [X_0, Y_0] \xrightarrow{\kappa_i := \mathrm{id}} [X_0, Y_0]$
- For $\mathrm{deg}(i) = n + 1$, inductively get $[X, Y]$ defined on $\partial({}^{i}/\mathcal{I}) = {}^{i}/\mathcal{I} - \{\mathrm{id}_i\}$ and $\kappa_j \colon [X, Y]_j \longrightarrow [X_j, Y_j]$ for $\mathrm{id} \neq f \colon i \to j \in \partial({}^{i}/\mathcal{I})$



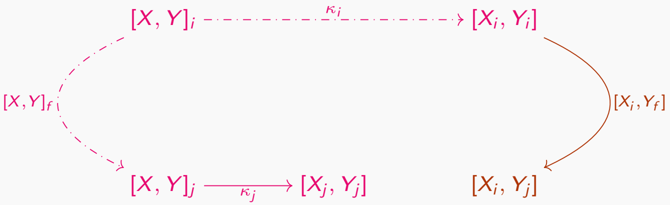
Inductively constructing diagram exponentials

Proposition

Let $(\mathcal{I}, \deg)$ be an inverse category and $\mathbb{C}$ be a cartesian closed category. Then, for diagrams $X, Y \in \mathbb{C}^{\mathcal{I}}$, the exponential diagram $[X, Y]$ is constructed by way of induction so that at each $i \in \mathcal{I}$, one has a comparison map

$$[X, Y]_i \xrightarrow{\kappa_i} [X_i, Y_i]$$

- For $\deg(i) = 0$, one puts $[X, Y]_0 \coloneqq [X_0, Y_0] \xrightarrow{\kappa_i := \mathrm{id}} [X_0, Y_0]$
- For $\deg(i) = n + 1$, inductively get $[X, Y]$ defined on $\partial({}^{i}/\mathcal{I}) = {}^{i}/\mathcal{I} - \{\mathrm{id}_i\}$ and $\kappa_j \colon [X, Y]_j \longrightarrow [X_j, Y_j]$ for $\mathrm{id} \neq f \colon i \to j \in \partial({}^{i}/\mathcal{I})$



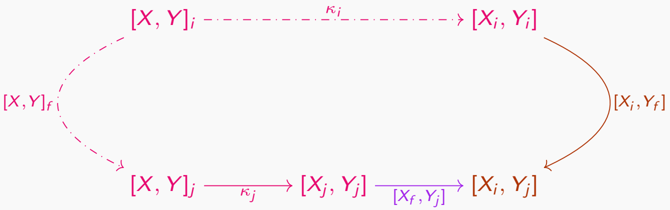
Inductively constructing diagram exponentials

Proposition

Let $(\mathcal{I}, \deg)$ be an inverse category and $\mathbb{C}$ be a cartesian closed category. Then, for diagrams $X, Y \in \mathbb{C}^{\mathcal{I}}$, the exponential diagram $[X, Y]$ is constructed by way of induction so that at each $i \in \mathcal{I}$, one has a comparison map

$$[X, Y]_i \xrightarrow{\kappa_i} [X_i, Y_i]$$

- For $\deg(i) = 0$, one puts $[X, Y]_0 := [X_0, Y_0] \xrightarrow{\kappa_i := \mathrm{id}} [X_0, Y_0]$
- For $\deg(i) = n + 1$, inductively get $[X, Y]$ defined on $\partial({}^{i}/\mathcal{I}) = {}^{i}/\mathcal{I} - \{\mathrm{id}_i\}$ and $\kappa_j : [X, Y]_j \longrightarrow [X_j, Y_j]$ for $\mathrm{id} \neq f : i \to j \in \partial({}^{i}/\mathcal{I})$



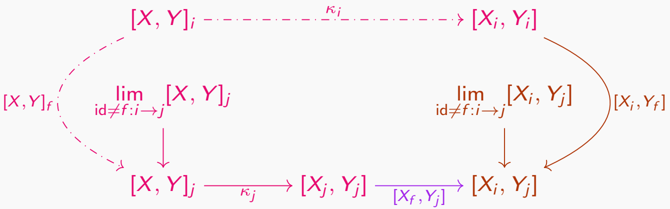
Inductively constructing diagram exponentials

Proposition

Let $(\mathcal{I}, \deg)$ be an inverse category and $\mathbb{C}$ be a cartesian closed category. Then, for diagrams $X, Y \in \mathbb{C}^{\mathcal{I}}$, the exponential diagram $[X, Y]$ is constructed by way of induction so that at each $i \in \mathcal{I}$, one has a comparison map

$$[X, Y]_i \xrightarrow{\kappa_i} [X_i, Y_i]$$

- For $\deg(i) = 0$, one puts $[X, Y]_0 \coloneqq [X_0, Y_0] \xrightarrow{\kappa_i := \mathrm{id}} [X_0, Y_0]$
- For $\deg(i) = n + 1$, inductively get $[X, Y]$ defined on $\partial({}^{i}/\mathcal{I}) = {}^{i}/\mathcal{I} - \{\mathrm{id}_i\}$ and $\kappa_j \colon [X, Y]_j \longrightarrow [X_j, Y_j]$ for $\mathrm{id} \neq f \colon i \to j \in \partial({}^{i}/\mathcal{I})$

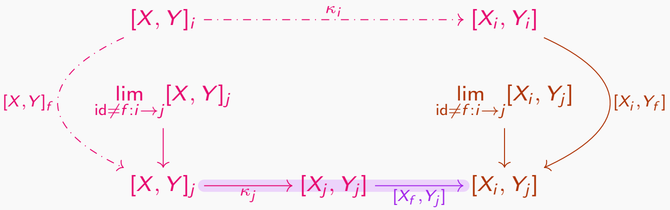


Proposition

Let $(\mathcal{I}, \deg)$ be an inverse category and $\mathbb{C}$ be a cartesian closed category. Then, for diagrams $X, Y \in \mathbb{C}^{\mathcal{I}}$, the exponential diagram $[X, Y]$ is constructed by way of induction so that at each $i \in \mathcal{I}$, one has a comparison map

$$[X, Y]_i \xrightarrow{\kappa_i} [X_i, Y_i]$$

- For $\deg(i) = 0$, one puts $[X, Y]_0 \coloneqq [X_0, Y_0] \xrightarrow{\kappa_i \coloneqq \mathrm{id}} [X_0, Y_0]$
- For $\deg(i) = n + 1$, inductively get $[X, Y]$ defined on $\partial({}^i/\mathcal{I}) = {}^i/\mathcal{I} - \{\mathrm{id}_i\}$ and $\kappa_j \colon [X, Y]_j \longrightarrow [X_j, Y_j]$ for $\mathrm{id} \neq f \colon i \to j \in \partial({}^i/\mathcal{I})$



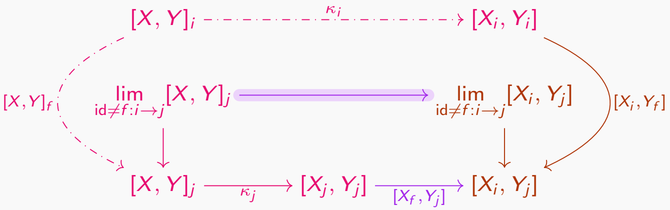
Inductively constructing diagram exponentials

Proposition

Let $(\mathcal{I}, \deg)$ be an inverse category and $\mathbb{C}$ be a cartesian closed category. Then, for diagrams $X, Y \in \mathbb{C}^{\mathcal{I}}$, the exponential diagram $[X, Y]$ is constructed by way of induction so that at each $i \in \mathcal{I}$, one has a comparison map

$$[X, Y]_i \xrightarrow{\kappa_i} [X_i, Y_i]$$

- For $\deg(i) = 0$, one puts $[X, Y]_0 \coloneqq [X_0, Y_0] \xrightarrow{\kappa_i := \mathrm{id}} [X_0, Y_0]$
- For $\deg(i) = n + 1$, inductively get $[X, Y]$ defined on $\partial({}^{i}/\mathcal{I}) = {}^{i}/\mathcal{I} - \{\mathrm{id}_i\}$ and $\kappa_j \colon [X, Y]_j \longrightarrow [X_j, Y_j]$ for $\mathrm{id} \neq f \colon i \to j \in \partial({}^{i}/\mathcal{I})$



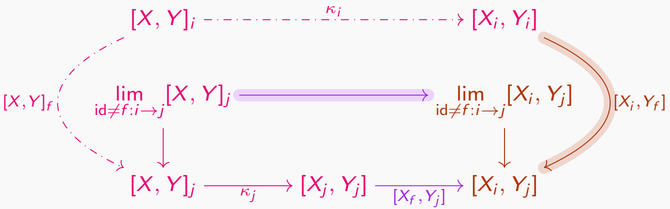
Inductively constructing diagram exponentials

Proposition

Let $(\mathcal{I}, \deg)$ be an inverse category and $\mathbb{C}$ be a cartesian closed category. Then, for diagrams $X, Y \in \mathbb{C}^{\mathcal{I}}$, the exponential diagram $[X, Y]$ is constructed by way of induction so that at each $i \in \mathcal{I}$, one has a comparison map

$$[X, Y]_i \xrightarrow{\kappa_i} [X_i, Y_i]$$

- For $\deg(i) = 0$, one puts $[X, Y]_0 \coloneqq [X_0, Y_0] \xrightarrow{\kappa_i := \mathrm{id}} [X_0, Y_0]$
- For $\deg(i) = n + 1$, inductively get $[X, Y]$ defined on $\partial({}^{i}/\mathcal{I}) = {}^{i}/\mathcal{I} - \{\mathrm{id}_i\}$ and $\kappa_j \colon [X, Y]_j \longrightarrow [X_j, Y_j]$ for $\mathrm{id} \neq f \colon i \to j \in \partial({}^{i}/\mathcal{I})$



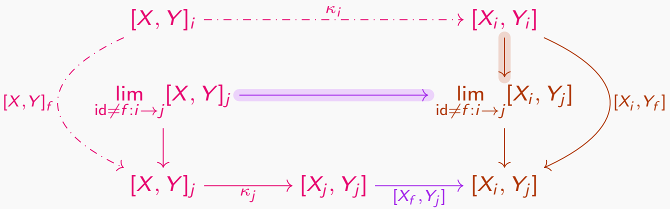
Inductively constructing diagram exponentials

Proposition

Let $(\mathcal{I}, \deg)$ be an inverse category and $\mathbb{C}$ be a cartesian closed category. Then, for diagrams $X, Y \in \mathbb{C}^{\mathcal{I}}$, the exponential diagram $[X, Y]$ is constructed by way of induction so that at each $i \in \mathcal{I}$, one has a comparison map

$$[X,Y]_i \xrightarrow{\kappa_i} [X_i, Y_i]$$

- For $\deg(i) = 0$, one puts $[X,Y]_0 \coloneqq [X_0, Y_0] \xrightarrow{\kappa_i := \mathrm{id}} [X_0, Y_0]$
- For $\deg(i) = n+1$, inductively get $[X,Y]$ defined on $\partial({}^{i}/\mathcal{I}) = {}^{i}/\mathcal{I} - \{\mathrm{id}_i\}$ and $\kappa_j \colon [X,Y]_j \longrightarrow [X_j, Y_j]$ for $\mathrm{id} \neq f \colon i \to j \in \partial({}^{i}/\mathcal{I})$



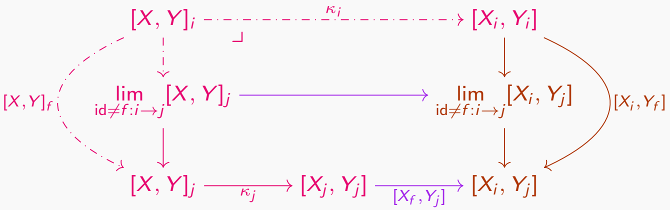
Inductively constructing diagram exponentials

Proposition

Let $(\mathcal{I}, \mathsf{deg})$ be an inverse category and $\mathbb{C}$ be a cartesian closed category. Then, for diagrams $X, Y \in \mathbb{C}^{\mathcal{I}}$, the exponential diagram $[X, Y]$ is constructed by way of induction so that at each $i \in \mathcal{I}$, one has a comparison map

$$[X, Y]_i \xrightarrow{\kappa_i} [X_i, Y_i]$$

- For $\mathsf{deg}(i) = 0$, one puts $[X, Y]_0 := [X_0, Y_0] \xrightarrow{\kappa_i := \mathrm{id}} [X_0, Y_0]$
- For $\mathsf{deg}(i) = n + 1$, inductively get $[X, Y]$ defined on $\partial({}^{i}/\mathcal{I}) = {}^{i}/\mathcal{I} - \{\mathrm{id}_i\}$ and $\kappa_j \colon [X, Y]_j \longrightarrow [X_j, Y_j]$ for $\mathrm{id} \neq f \colon i \to j \in \partial({}^{i}/\mathcal{I})$



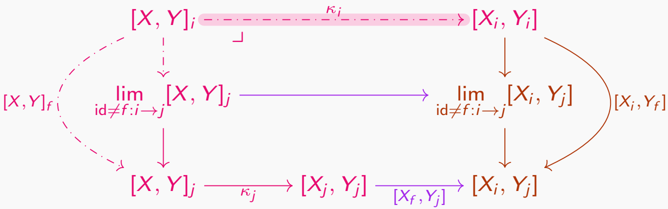
Inductively constructing diagram exponentials

Proposition

Let $(\mathcal{I}, \deg)$ be an inverse category and $\mathbb{C}$ be a cartesian closed category. Then, for diagrams $X, Y \in \mathbb{C}^{\mathcal{I}}$, the exponential diagram $[X, Y]$ is constructed by way of induction so that at each $i \in \mathcal{I}$, one has a comparison map

$$[X, Y]_i \xrightarrow{\kappa_i} [X_i, Y_i]$$

- For $\deg(i) = 0$, one puts $[X, Y]_0 \coloneqq [X_0, Y_0] \xrightarrow{\kappa_i := \mathrm{id}} [X_0, Y_0]$
- For $\deg(i) = n + 1$, inductively get $[X, Y]$ defined on $\partial({}^{i}/\mathcal{I}) = {}^{i}/\mathcal{I} - \{\mathrm{id}_i\}$ and $\kappa_j \colon [X, Y]_j \longrightarrow [X_j, Y_j]$ for $\mathrm{id} \neq f \colon i \to j \in \partial({}^{i}/\mathcal{I})$



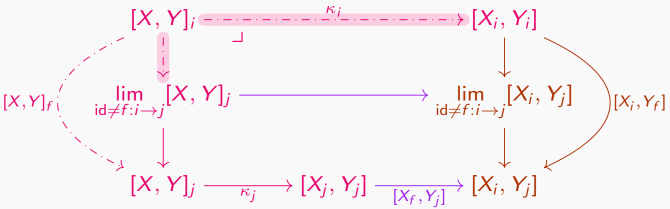
Inductively constructing diagram exponentials

Proposition

Let $(\mathcal{I}, \deg)$ be an inverse category and $\mathbb{C}$ be a cartesian closed category. Then, for diagrams $X, Y \in \mathbb{C}^{\mathcal{I}}$, the exponential diagram $[X, Y]$ is constructed by way of induction so that at each $i \in \mathcal{I}$, one has a comparison map

$$[X, Y]_i \xrightarrow{\kappa_i} [X_i, Y_i]$$

- For $\deg(i) = 0$, one puts $[X, Y]_0 \coloneqq [X_0, Y_0] \xrightarrow{\kappa_i := \mathrm{id}} [X_0, Y_0]$
- For $\deg(i) = n + 1$, inductively get $[X, Y]$ defined on $\partial({}^{i}/\mathcal{I}) = {}^{i}/\mathcal{I} - \{\mathrm{id}_i\}$ and $\kappa_j \colon [X, Y]_j \longrightarrow [X_j, Y_j]$ for $\mathrm{id} \neq f \colon i \to j \in \partial({}^{i}/\mathcal{I})$



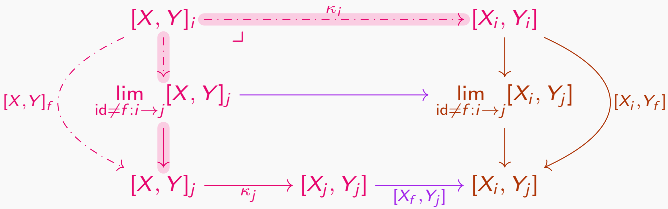
Inductively constructing diagram exponentials

Proposition

Let $(\mathcal{I}, \deg)$ be an inverse category and $\mathbb{C}$ be a cartesian closed category. Then, for diagrams $X, Y \in \mathbb{C}^{\mathcal{I}}$, the exponential diagram $[X, Y]$ is constructed by way of induction so that at each $i \in \mathcal{I}$, one has a comparison map

$$[X, Y]_i \xrightarrow{\kappa_i} [X_i, Y_i]$$

- For $\deg(i) = 0$, one puts $[X, Y]_0 \coloneqq [X_0, Y_0] \xrightarrow{\kappa_i := \mathrm{id}} [X_0, Y_0]$
- For $\deg(i) = n + 1$, inductively get $[X, Y]$ defined on $\partial({}^i/\mathcal{I}) = {}^i/\mathcal{I} - \{\mathrm{id}_i\}$ and $\kappa_j \colon [X, Y]_j \longrightarrow [X_j, Y_j]$ for $\mathrm{id} \neq f \colon i \to j \in \partial({}^i/\mathcal{I})$



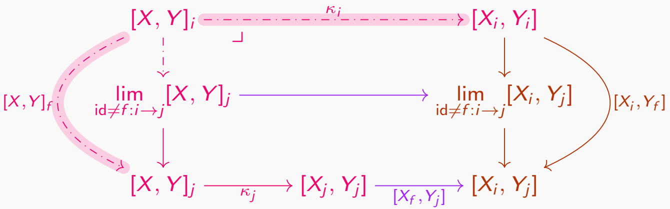
Inductively constructing diagram exponentials

Proposition

Let $(\mathcal{I}, \deg)$ be an inverse category and $\mathbb{C}$ be a cartesian closed category. Then, for diagrams $X, Y \in \mathbb{C}^{\mathcal{I}}$, the exponential diagram $[X, Y]$ is constructed by way of induction so that at each $i \in \mathcal{I}$, one has a comparison map

$$[X, Y]_i \xrightarrow{\kappa_i} [X_i, Y_i]$$

- For $\deg(i) = 0$, one puts $[X, Y]_0 \coloneqq [X_0, Y_0] \xrightarrow{\kappa_i := \mathrm{id}} [X_0, Y_0]$
- For $\deg(i) = n + 1$, inductively get $[X, Y]$ defined on $\partial({}^{i}/\mathcal{I}) = {}^{i}/\mathcal{I} - \{\mathrm{id}_i\}$ and $\kappa_j \colon [X, Y]_j \longrightarrow [X_j, Y_j]$ for $\mathrm{id} \neq f \colon i \to j \in \partial({}^{i}/\mathcal{I})$



Assumption

- Weak equivalences of $\mathbb{C}$ are stable under pullback along fibrations

Exponentials in inverse homotopical diagrams

Assumption

- Weak equivalences of $\mathbb{C}$ are stable under pullback along fibrations
- Limits of pointwise weakly equivalent Reedy fibrant $\mathbb{C}$-valued inverse diagrams is again weakly equivalent

Assumption

- Weak equivalences of $\mathbb{C}$ are stable under pullback along fibrations
- Limits of pointwise weakly equivalent Reedy fibrant $\mathbb{C}$-valued inverse diagrams is again weakly equivalent
- Trivial fibrations (i.e. fibrations that are also weak equivalences) are pullback-stable

Exponentials in inverse homotopical diagrams

Assumption

- Weak equivalences of $\mathbb{C}$ are stable under pullback along fibrations
- Limits of pointwise weakly equivalent Reedy fibrant $\mathbb{C}$-valued inverse diagrams is again weakly equivalent
- Trivial fibrations (i.e. fibrations that are also weak equivalences) are pullback-stable
- $[-,-]\colon \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathbb{C}$ preserves weak equivalences in both variables and fibrations in the second argument

Exponentials in inverse homotopical diagrams

Assumption

- Weak equivalences of $\mathbb{C}$ are stable under pullback along fibrations
- Limits of pointwise weakly equivalent Reedy fibrant $\mathbb{C}$-valued inverse diagrams is again weakly equivalent
- Trivial fibrations (i.e. fibrations that are also weak equivalences) are pullback-stable
- $[-,-]\colon \mathbb{C}^{\mathrm{op}} \times \mathbb{C} \to \mathbb{C}$ preserves weak equivalences in both variables and fibrations in the second argument

Example.

- Models of intensional type theory such as universe categories, π-tribes

Exponentials in inverse homotopical diagrams

Assumption

- Weak equivalences of $\mathbb{C}$ are stable under pullback along fibrations
- Limits of pointwise weakly equivalent Reedy fibrant $\mathbb{C}$-valued inverse diagrams is again weakly equivalent
- Trivial fibrations (i.e. fibrations that are also weak equivalences) are pullback-stable
- $[-,-]\colon \mathbb{C}^{\mathrm{op}} \times \mathbb{C} \to \mathbb{C}$ preserves weak equivalences in both variables and fibrations in the second argument

Example.

- Models of intensional type theory such as universe categories, π-tribes
- Right proper Cisinski model structures such as simplicial sets, cubical sets

Exponentials in inverse homotopical diagrams

Assumption

- Weak equivalences of $\mathbb{C}$ are stable under pullback along fibrations
- Limits of pointwise weakly equivalent Reedy fibrant $\mathbb{C}$-valued inverse diagrams is again weakly equivalent
- Trivial fibrations (i.e. fibrations that are also weak equivalences) are pullback-stable
- $[-,-]\colon \mathbb{C}^{\mathrm{op}} \times \mathbb{C} \to \mathbb{C}$ preserves weak equivalences in both variables and fibrations in the second argument

Lemma (Underslice version of Kapulkin–Lumsdaine 2021)

For all $i \in \mathcal{I}$ if Reedy fibrant diagrams $X, Y \in \mathbb{C}^{\mathcal{I}}$ sends all maps under i to weak equivalences then so does $[X, Y] \in \mathbb{C}^{\mathcal{I}}$.

Exponentials in inverse homotopical diagrams

Assumption

- Weak equivalences of $\mathbb{C}$ are stable under pullback along fibrations
- Limits of pointwise weakly equivalent Reedy fibrant $\mathbb{C}$-valued inverse diagrams is again weakly equivalent
- Trivial fibrations (i.e. fibrations that are also weak equivalences) are pullback-stable
- $[-,-]\colon \mathbb{C}^{\mathrm{op}} \times \mathbb{C} \to \mathbb{C}$ preserves weak equivalences in both variables and fibrations in the second argument

Lemma (Underslice version of Kapulkin–Lumsdaine 2021)

For all $i \in \mathcal{I}$ if Reedy fibrant diagrams $X, Y \in \mathbb{C}^{\mathcal{I}}$ sends all maps under i to weak equivalences then so does $[X, Y] \in \mathbb{C}^{\mathcal{I}}$.

Lemma

If $X, Y \in \mathbb{C}^{\mathcal{I}}$ sends the initial object $f\colon i \to j \in \partial({}^{i}/\mathcal{I}) = {}^{i}/\mathcal{I} - \{\mathrm{id}\}$ to a weak equivalence then so does $[X, Y] \in \mathbb{C}^{\mathcal{I}}$.

Exponentials in inverse homotopical diagrams

Assumption

- Weak equivalences of $\mathbb{C}$ are stable under pullback along fibrations
- Limits of pointwise weakly equivalent Reedy fibrant $\mathbb{C}$-valued inverse diagrams is again weakly equivalent
- Trivial fibrations (i.e. fibrations that are also weak equivalences) are pullback-stable
- $[-,-]\colon \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathbb{C}$ preserves weak equivalences in both variables and fibrations in the second argument

Theorem

In this case, when Reedy fibrant $X, Y \in \mathbb{C}^{\mathcal{I}}$ are homotopical, so is $[X, Y] \in \mathbb{C}^{\mathcal{I}}$.

Exponentials in inverse homotopical diagrams

Assumption

- Weak equivalences of $\mathbb{C}$ are stable under pullback along fibrations
- Limits of pointwise weakly equivalent Reedy fibrant $\mathbb{C}$-valued inverse diagrams is again weakly equivalent
- Trivial fibrations (i.e. fibrations that are also weak equivalences) are pullback-stable
- $[-,-]\colon \mathbb{C}^{\mathrm{op}} \times \mathbb{C} \to \mathbb{C}$ preserves weak equivalences in both variables and fibrations in the second argument

Theorem

Suppose for each $i \in \mathcal{I}$, if there is a weak equivalence $\mathrm{id} \neq w\colon i \to j$ then:

In this case, when Reedy fibrant $X, Y \in \mathbb{C}^{\mathcal{I}}$ are homotopical, so is $[X, Y] \in \mathbb{C}^{\mathcal{I}}$.

Exponentials in inverse homotopical diagrams

Assumption

- Weak equivalences of $\mathbb{C}$ are stable under pullback along fibrations
- Limits of pointwise weakly equivalent Reedy fibrant $\mathbb{C}$-valued inverse diagrams is again weakly equivalent
- Trivial fibrations (i.e. fibrations that are also weak equivalences) are pullback-stable
- $[-,-]\colon \mathbb{C}^{\mathrm{op}} \times \mathbb{C} \to \mathbb{C}$ preserves weak equivalences in both variables and fibrations in the second argument

Theorem

Suppose for each $i \in \mathcal{I}$, if there is a weak equivalence $\mathrm{id} \neq w\colon i \to j$ then:

- either all $i \to k$ are weak equivalences;

In this case, when Reedy fibrant $X, Y \in \mathbb{C}^{\mathcal{I}}$ are homotopical, so is $[X, Y] \in \mathbb{C}^{\mathcal{I}}$.

Exponentials in inverse homotopical diagrams

Assumption

- Weak equivalences of $\mathbb{C}$ are stable under pullback along fibrations
- Limits of pointwise weakly equivalent Reedy fibrant $\mathbb{C}$-valued inverse diagrams is again weakly equivalent
- Trivial fibrations (i.e. fibrations that are also weak equivalences) are pullback-stable
- $[-,-]\colon \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathbb{C}$ preserves weak equivalences in both variables and fibrations in the second argument

Theorem

Suppose for each $i \in \mathcal{I}$, if there is a weak equivalence $\mathrm{id} \neq w\colon i \to j$ then:

- either all $i \to k$ are weak equivalences;
- or all $(i \to k) \neq \mathrm{id}$ factors uniquely through $w\colon i \to j$

In this case, when Reedy fibrant $X, Y \in \mathbb{C}^{\mathcal{I}}$ are homotopical, so is $[X, Y] \in \mathbb{C}^{\mathcal{I}}$.

Assumption

- Weak equivalences of $\mathbb{C}$ are stable under pullback along fibrations
- Limits of pointwise weakly equivalent Reedy fibrant $\mathbb{C}$-valued inverse diagrams is again weakly equivalent
- Trivial fibrations (i.e. fibrations that are also weak equivalences) are pullback-stable

Pushforwards in inverse homotopical diagrams

Assumption

- Weak equivalences of $\mathbb{C}$ are stable under pullback along fibrations
- Limits of pointwise weakly equivalent Reedy fibrant $\mathbb{C}$-valued inverse diagrams is again weakly equivalent
- Trivial fibrations (i.e. fibrations that are also weak equivalences) are pullback-stable
- Pushforwards along fibrations preserve weak equivalences

Pushforwards in inverse homotopical diagrams

Assumption

- Weak equivalences of $\mathbb{C}$ are stable under pullback along fibrations
- Limits of pointwise weakly equivalent Reedy fibrant $\mathbb{C}$-valued inverse diagrams is again weakly equivalent
- Trivial fibrations (i.e. fibrations that are also weak equivalences) are pullback-stable
- Pushforwards along fibrations preserve weak equivalences if f, g are weak equivalences

$$\begin{array}{ccc} B' & \xrightarrow{p'} & A' \\ {\scriptstyle g}\downarrow{\scriptstyle \sim} & & {\scriptstyle \sim}\downarrow{\scriptstyle f} \\ B & \xrightarrow[p]{} & A \end{array}$$

Pushforwards in inverse homotopical diagrams

Assumption

- Weak equivalences of $\mathbb{C}$ are stable under pullback along fibrations
- Limits of pointwise weakly equivalent Reedy fibrant $\mathbb{C}$-valued inverse diagrams is again weakly equivalent
- Trivial fibrations (i.e. fibrations that are also weak equivalences) are pullback-stable
- Pushforwards along fibrations preserve weak equivalences and the pullback-pushforward mate $f^* p_* \to p'_* g^*$ is a weak equivalence if f, g are weak equivalences

Pushforwards in inverse homotopical diagrams

Assumption

- Weak equivalences of $\mathbb{C}$ are stable under pullback along fibrations
- Limits of pointwise weakly equivalent Reedy fibrant $\mathbb{C}$-valued inverse diagrams is again weakly equivalent
- Trivial fibrations (i.e. fibrations that are also weak equivalences) are pullback-stable
- Pushforwards along fibrations preserve weak equivalences and the pullback-pushforward mate $f^*p_* \to p'_*g^*$ is a weak equivalence if f, g are weak equivalences

$$\begin{array}{ccc} B' & \xrightarrow{p'} & A' \\ {\scriptstyle g}\downarrow{\scriptstyle\sim} & & {\scriptstyle\sim}\downarrow{\scriptstyle f} \\ B & \xrightarrow[p]{} & A \end{array} \quad \Longrightarrow \quad \begin{array}{ccc} \mathbb{C}/B' & \xrightarrow{p'_*} & \mathbb{C}/A' \\ {\scriptstyle g^*}\uparrow & \overset{\sim}{\Longleftarrow} & \uparrow{\scriptstyle f^*} \\ \mathbb{C}/B & \xrightarrow[p_*]{} & \mathbb{C}/A \end{array}$$

Example.

- Models of intensional type theory such as universe categories, π-tribes

Pushforwards in inverse homotopical diagrams

Assumption

- Weak equivalences of $\mathbb{C}$ are stable under pullback along fibrations
- Limits of pointwise weakly equivalent Reedy fibrant $\mathbb{C}$-valued inverse diagrams is again weakly equivalent
- Trivial fibrations (i.e. fibrations that are also weak equivalences) are pullback-stable
- Pushforwards along fibrations preserve weak equivalences and the pullback-pushforward mate $f^* p_* \to p'_* g^*$ is a weak equivalence if f, g are weak equivalences

$$\begin{array}{ccc} B' & \xrightarrow{p'} & A' \\ {\scriptstyle g}\downarrow{\scriptstyle \sim} & & {\scriptstyle \sim}\downarrow{\scriptstyle f} \\ B & \xrightarrow[p]{} & A \end{array} \quad \Longrightarrow \quad \begin{array}{ccc} \mathbb{C}/B' & \xrightarrow{p'_*} & \mathbb{C}/A' \\ {\scriptstyle g^*}\uparrow & \overset{\sim}{\nwarrow} & \uparrow{\scriptstyle f^*} \\ \mathbb{C}/B & \xrightarrow[p_*]{} & \mathbb{C}/A \end{array}$$

Example.

- Models of intensional type theory such as universe categories, π-tribes
- Right proper Cisinski model structures such as simplicial sets, cubical sets

Pushforwards in inverse homotopical diagrams

Assumption

- Weak equivalences of $\mathbb{C}$ are stable under pullback along fibrations
- Limits of pointwise weakly equivalent Reedy fibrant $\mathbb{C}$-valued inverse diagrams is again weakly equivalent
- Trivial fibrations (i.e. fibrations that are also weak equivalences) are pullback-stable
- Pushforwards along fibrations preserve weak equivalences and the pullback-pushforward mate $f^* p_* \to p'_* g^*$ is a weak equivalence if f, g are weak equivalences

$$\begin{array}{ccc} B' & \xrightarrow{p'} & A' \\ g \downarrow \sim & & \sim \downarrow f \\ B & \xrightarrow[p]{} & A \end{array} \quad \Longrightarrow \quad \begin{array}{ccc} \mathbb{C}/B' & \xrightarrow{p'_*} & \mathbb{C}/A' \\ g^* \uparrow & \overset{\sim}{\Leftarrow} & \uparrow f^* \\ \mathbb{C}/B & \xrightarrow[p_*]{} & \mathbb{C}/A \end{array}$$

Theorem

Suppose for each $i \in \mathcal{I}$, if $\mathrm{id} \neq w\colon i \to j$ is a weak equivalence then

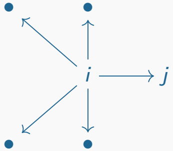
all $i \to k$ are weak equivalences or all $(i \to k) \neq \mathrm{id}$ factors uniquely via w

Homotopical Reedy fibrations are closed under pushforwards along homotopical Reedy fibrations.

Shape of $\mathcal{I}$

For each $i \in \mathcal{I}$, if $\mathrm{id} \neq w \colon i \to j$ is a weak equivalence then

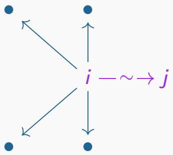
- all $i \to k$ are weak equivalences



Shape of $\mathcal{I}$

For each $i \in \mathcal{I}$, if $\mathrm{id} \neq w \colon i \to j$ is a weak equivalence then

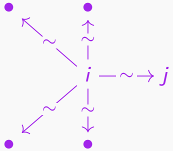
- all $i \to k$ are weak equivalences



Shape of $\mathcal{I}$

For each $i \in \mathcal{I}$, if $\mathrm{id} \neq w \colon i \to j$ is a weak equivalence then

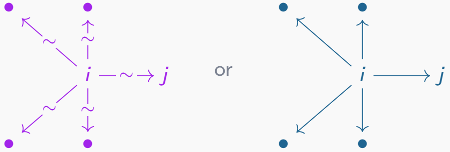
- all $i \to k$ are weak equivalences



Shape of $\mathcal{I}$

For each $i \in \mathcal{I}$, if $\mathrm{id} \neq w\colon i \to j$ is a weak equivalence then

- all $i \to k$ are weak equivalences; or



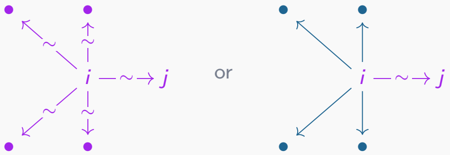
- all $(i \to k) \neq \mathrm{id}$ factors uniquely through $w\colon i \to j$

Pushforwards in inverse homotopical diagrams

Shape of $\mathcal{I}$

For each $i \in \mathcal{I}$, if $\mathrm{id} \neq w\colon i \to j$ is a weak equivalence then

- all $i \to k$ are weak equivalences; or



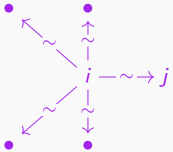
- all $(i \to k) \neq \mathrm{id}$ factors uniquely through $w\colon i \to j$

Pushforwards in inverse homotopical diagrams

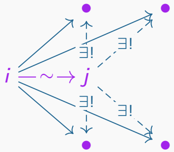
Shape of $\mathcal{I}$

For each $i \in \mathcal{I}$, if $\mathrm{id} \neq w \colon i \to j$ is a weak equivalence then

- all $i \to k$ are weak equivalences; or



or



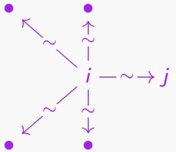
- all $(i \to k) \neq \mathrm{id}$ factors uniquely through $w \colon i \to j$

Pushforwards in inverse homotopical diagrams

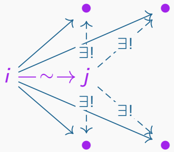
Shape of $\mathcal{I}$

For each $i \in \mathcal{I}$, if $\mathrm{id} \neq w \colon i \to j$ is a weak equivalence then

- all $i \to k$ are weak equivalences; or



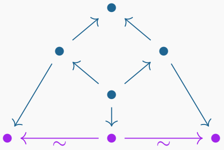
or



- all $(i \to k) \neq \mathrm{id}$ factors uniquely through $w \colon i \to j$

Example.

Valid shapes of $\mathcal{I}$

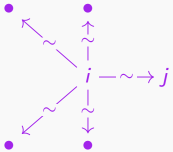


Pushforwards in inverse homotopical diagrams

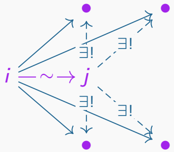
Shape of $\mathcal{I}$

For each $i \in \mathcal{I}$, if $\mathrm{id} \neq w \colon i \to j$ is a weak equivalence then

- all $i \to k$ are weak equivalences; or

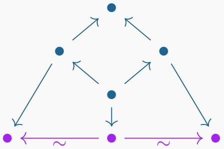


or



- all $(i \to k) \neq \mathrm{id}$ factors uniquely through $w \colon i \to j$

Example.

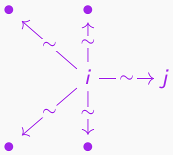
Valid shapes of $\mathcal{I}$	Invalid shapes of $\mathcal{I}$
	$0 \longleftarrow 01 \xrightarrow{\sim} 1$

Pushforwards in inverse homotopical diagrams

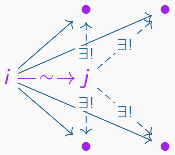
Shape of $\mathcal{I}$

For each $i \in \mathcal{I}$, if $\mathrm{id} \neq w\colon i \to j$ is a weak equivalence then

- all $i \to k$ are weak equivalences; or



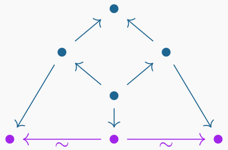
or



- all $(i \to k) \neq \mathrm{id}$ factors uniquely through $w\colon i \to j$

Example.

Valid shapes of $\mathcal{I}$ | Invalid shapes of $\mathcal{I}$



$$0 \longleftarrow 01 \xrightarrow{\sim} 1$$

Thank you for your attention

∼ ♡ ∼