Univalent Multicategories

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Multicategories

A classical presentation

A multicategory M is given by [1]

- A **collection** M_0 of *objects*.
- A **set** $M\left(\vec{A}, B\right)$ between $\vec{A}$: List M_0 and $B : M_0$ of *morphisms*.
- An associative operation $\circ$, where if $f : \mathrm{Hom}\left(\vec{B}, C\right)$ and $g_i : \mathrm{Hom}\left(\vec{A}^i, B_i\right)$,

$$f \circ [g_1, g_2, ..., g_n] : \mathrm{Hom}\left(\vec{A}^0 +\!\!+ \vec{A}^1 +\!\!+ ... +\!\!+ \vec{A}^n \;, C\right).$$

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and morphisms

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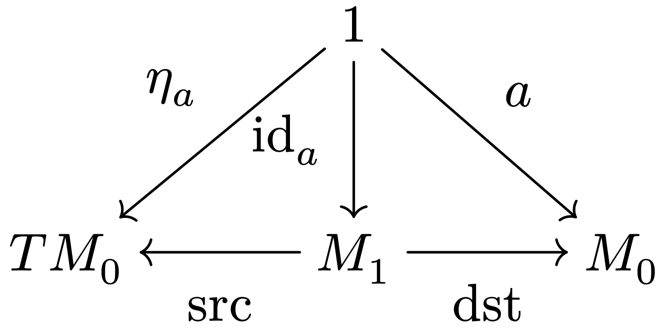
- An identity $\mathrm{id}_A : \mathrm{Hom}([A], A)$ such that

$$f \circ \left[\mathrm{id}_{A_1}, \mathrm{id}_{A_2}, ..., \mathrm{id}_{A_n}\right] = f = \mathrm{id}_B \circ [f].$$

Classical definitions

- Monads in the bicategory of T-spans for a cartesian monad T



Examples

- Multilinear maps
  - objects are R-modules
  - morphisms from $\overrightarrow{M}$ to N are functions $f : \prod_i M_i \to N$ such that

    $$f(x_1, x_2, ..., rx_i + y_i, ..., x_n) = rf(x_1, x_2, ..., x_i, ..., x_n) + f(x_1, x_2, ..., y_i, ..., x_n)$$
  - $\mathrm{id} : N \to N$ is the identity function
- Sequents
  - objects are lists Γ of types
  - morpisms from Γ to T are sequents $\Gamma \vdash x : T$

Distribution Types

The category Dist has

- $\mathrm{Ob} = \mathbb{N}$
- $\mathrm{Hom}(n, m) = \sum_{\substack{f \colon \mathrm{Fin}\ n \\ \to \mathrm{Maybe}\ (\mathrm{Fin}\ m)}} \underbrace{\prod_{j,k : \mathrm{Fin}\ n} (j < k) \to (fj \neq *) \to (fk \neq *) \to fj < fk}_{f \text{ partially ascending}}$

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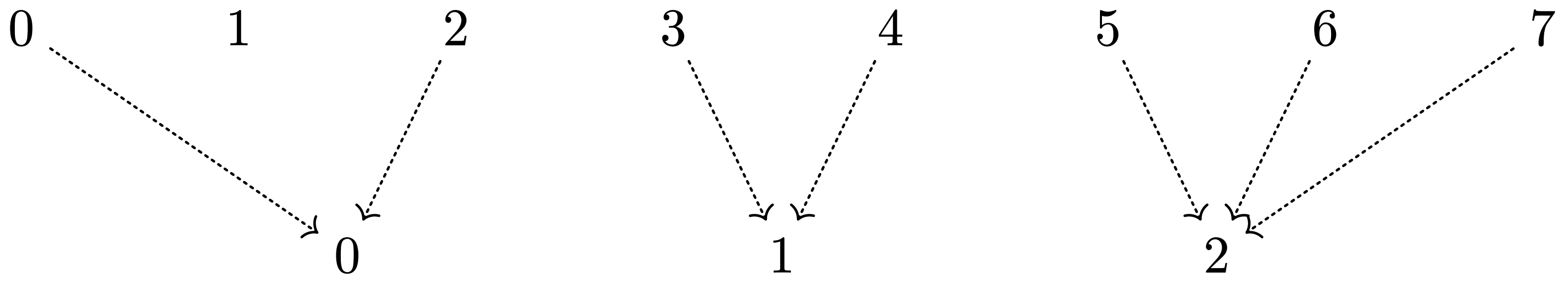

$f : \langle 8 \rangle \to \langle 3 \rangle$

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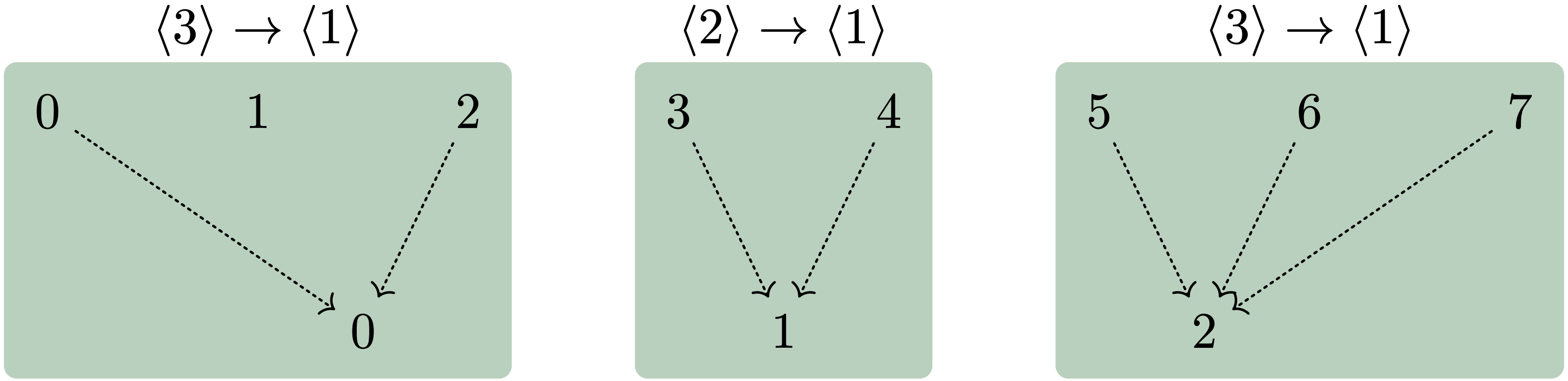

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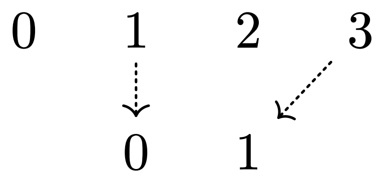
Inert and Active morphisms

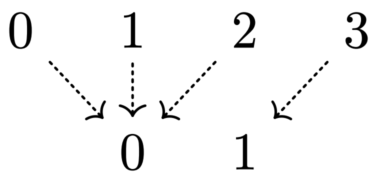
Inert



$f : \langle 4 \rangle \to \langle 2 \rangle$ inert

- partial bijections where $f : \langle n \rangle \to \langle m \rangle$ is defined n-times.
- represent how an m-ary vector can be contained within an n-ary vector.
- e.g. $\rho_i : \langle n \rangle \to \langle 1 \rangle$ such that $\rho_i : i \mapsto 0$

Active



$g : \langle 4 \rangle \to \langle 2 \rangle$ active

- totally defined functions
- $f : \langle n \rangle \to \langle m \rangle$ represents an multi-morphism from n to m with all variables “used”.
- e.g. the constant function $\Delta_0 : \langle n \rangle \to \langle 1 \rangle$

The Lore

Introduced by F.E.J. Linton in 1970!

By a $\underline{\mathcal{V}\text{-multimap}}$ $f = \langle f_1, \ldots, f_k \rangle : \langle A_1, \ldots, A_n \rangle \longrightarrow \langle B_1, \ldots, B_k \rangle$ of distribution type α ($\alpha : \{1 \ldots n\} \to \{1 \ldots k\}$ an order preserving function) we mean simply a sequence $f = \langle f_1, \ldots, f_k \rangle$ of multilinear $\mathcal{V}$-maps

$$f_i : \langle (A_j)_{j \in \alpha^{-1}(i)} \rangle \longrightarrow B_i \qquad (1 \leq i \leq k).$$

Multicategories (a second approach)

Displayed Categories

A category E *displayed over* C [2] has

- for each object $a : C$, a **collection** of objects E_a
- for $f : a \to b$, objects $A : E_a$, $B : E_b$ a **set** $E_f(A, B)$ of morphisms
- a *displayed composition* between

$$\circ : E_f(B, C) \times E_g(A, B) \to E_{f \circ g}(A, C)$$

- if $a : C$ and $A : E_a$, an identity $\mathrm{id}_A : E_{\mathrm{id}_a}(A, A)$
- **Dependent paths**

$$\lambda' : F \circ \mathrm{id}_A \underset{\lambda f}{=} F \qquad \rho' : \mathrm{id}_B \circ F \underset{\rho f}{=} F \qquad \alpha' : F \circ (G \circ H) \underset{\alpha f g h}{=} (F \circ G) \circ H$$

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$E_{f \circ \mathrm{id}_a}(A, B)$

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$F \circ \mathrm{id}_A$: $E_{f \circ \mathrm{id}_a}(A, B)$ $\qquad$ F : $E_f(A, B)$

Our definition

Definition 1 (Multicategory) A *multicategory* is a category M displayed over Dist.

…

- we call M_n a list of objects of length n
- given $F : M_f(A, B)$ we say F has *distribution type* f.

Important

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Note We have composition, identity and a collection M_1 of objects. What is missing?

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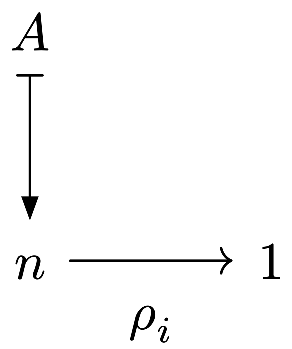

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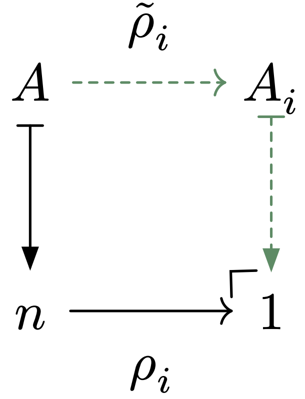

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Note Now we have projections on lists of objects and morphisms. Is that all?

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- with cocartesian lifts of all inert morphisms
- where $\mathrm{idx}(\cdot, \cdot)$ has inverse $\langle \ldots \rangle : \left(\prod_{i<n} \mathrm{Hom}_{\rho_i \circ f}(A, B_i)\right) \to \mathrm{Hom}_f(A, B)$
- an operation $\uparrow : M_1^n \to M_n$
  - with cocartesian morphisms $p_i : M_{\rho_i}(\uparrow A, A_i)$
  - *i.e.* $A_i \cong (\uparrow A)_i$

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🎉

an OFS on multicategories

Given a multicategory M,

- if $f : \langle n \rangle \to \langle m \rangle$ is inert, we call $F : M_f(A, B)$ *inert* if F is f-cocartesian.
- if $g : \langle n \rangle \to \langle m \rangle$ is active, all morphisms $M_g(A, B)$ are called *active*

Theorem (from Lurie [3]) Inert and active morphisms form an OFS on $\int M$.

Functors between multicategories

Definition 2 A *functor between multicategories* M and N consists of

- A functor $\mathcal{T} : M \Rightarrow N$ displayed over the identity functor on Dist
- such that $\mathcal{T}$ preserves inert morphisms

thus if $\mathcal{T} : M \to N$, $F : M_f(A, B)$

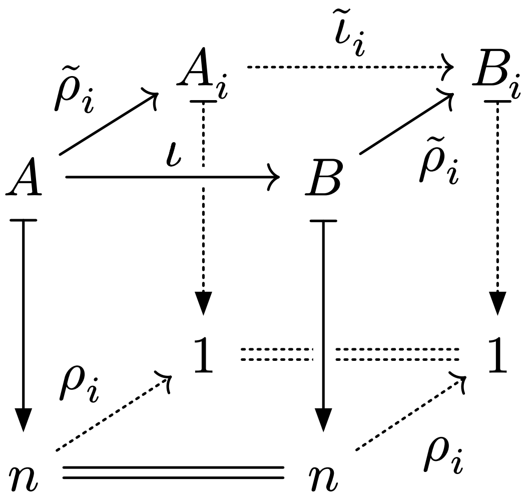
$$\mathcal{T}(A_j) \simeq \mathcal{T}(A)_j$$

and

$$\mathcal{T}(F_j) = \mathcal{T}(\tilde{\rho}_j \circ F) = \mathcal{T}(\tilde{\rho}_j) \circ \mathcal{T}(F) = \tilde{\rho}_j \circ \mathcal{T}(F) = \mathcal{T}(F)_j$$

Univalent Multicategories

- any isomorphism in a multicategory is of the form $M([A], A)$
- a displayed of the form $\iota : M_{\mathrm{id}_n}(A, B)$ (as Dist is univalent)



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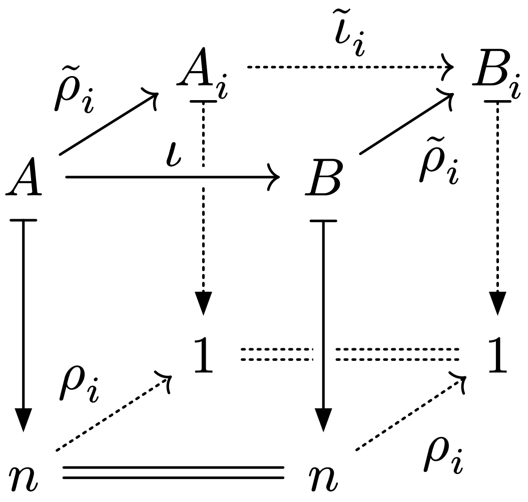

- thus we want every isomorphism in $M_{\mathrm{id}_n}(A, B)$ to give $A \underset{M_n}{=} B$
- this is exactly displayed univalence [2] over Dist

Definition 3 (Univalent Multicategory) A *univalent multicategory* is a multicategory M such that M is univalent (as a displayed category)

- if N is univalent, the category of functors $\mathbf{Multi}(M, N)$ is univalent
- if M is a univalent displayed category over Dist, then $\mathsf{is\_multicat}(M)$ is a mere proposition

∴ the bicategory Multi is univalent bicategory* [4].

*assuming equivalence is an identity system for categories displayed over a univalent base

Future Work

- Show that the “classical” definition of multicategories consides with the displayed definition.
- Prove global univalence for the bicategory of displayed categories over a shared base and vertical functors

Thank you!

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[1] T. Leinster, *Higher Operads, Higher Categories.* in London Mathematical Society Lecture Note Series. Cambridge University Press, 2004.

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