A general cubical framework for coherence theorems

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Motivation: conservativity conjectures for type theory

Hofmann 1995: **Extensional Type Theory** is conservative over **Intensional Type Theory** with Function Extensionality and Uniqueness of Identity Proofs

Conjecture(TYPES'17): HoTT is conservative over HoTT with "axiomatic" identity types.

$$\frac{\cdots \qquad d : P(x, \mathsf{refl})}{\mathsf{J}_\beta(d) : \mathsf{Id}(\mathsf{J}(d, \mathsf{refl}), d)} \qquad\qquad \frac{\cdots \qquad d : P(x, \mathsf{refl})}{\mathsf{J}(d, \mathsf{refl}) \equiv d}$$

Other conservativity conjectures:

- Computational HoTT is conservative over Axiomatic HoTT?
- HoTT + SProp + $(\text{SProp} \simeq \text{HProp})$ is conservative over HoTT?
- Definitional semiring structure on natural numbers?
- Definitional 1-groupoid structure on identity types?
- ...

Running example: strictification for monoidal categories

There is a free-forgetful adjunction between $\mathbf{MonCat}_w$ and $\mathbf{MonCat}_s$.

Strictification: For every cofibrant monoidal weak category $\mathcal{M}_w$, the unit

$$\eta_{\mathcal{M}} : \mathcal{M}_w \to \mathcal{M}_s$$

is a weak equivalence of monoidal categories.

Assume $(f : \mathsf{Hom}(x, y)) \in \mathcal{M}_w$ and $(g : \mathsf{Hom}(y', z)) \in \mathcal{M}_w$ such that $\eta(y) = \eta(y')$.

$$(\eta(g) \circ \eta(f)) \in \mathcal{M}_s \qquad \rightsquigarrow \qquad (f \circ e^*(g)) \in \mathcal{M}_w$$

where $e : y \cong y'$ and $e^* : \mathsf{Hom}(y', z) \to \mathsf{Hom}(y, z)$.

Need to make sure that the choice of e does not matter.
$\leadsto$ Only choose e among isomorphisms built from id, α, ρ, λ and $(- \otimes -)$.
$\leadsto$ Prove that two such parallel isomorphisms are equal (coherence).

Goal

Reduction of strictification/conservativity:

Every weak algebra is equivalent to a strict algebra.
The maps $\eta_{\mathcal{M}} : \mathcal{M}_w \to \mathcal{M}_s$ are weak equivalence.
Weak model structrures on $\mathrm{Alg}(\mathbb{T}_w)$ and $\mathrm{Alg}(\mathbb{T}_s)$ are Quillen equivalent.

to coherence:

Any diagram made of associators and unitors commute.
Any two identifications between terms are equal.
Some ∞-groupoids are 0-truncated.

General setup: generalized algebraic theories

- Generalized Algebraic Theories $\mathbb{T}_w$ and $\mathbb{T}_s$.
  Sorts $(\Gamma \vdash A)$, operations $(\Gamma \vdash f : A)$ and equations $(\Gamma \vdash x = y : A)$.

$$
\begin{aligned}
&\vdash \mathsf{Ob} \\
&x, y : \mathsf{Ob} \vdash \mathsf{Hom}(x, y) \\
&x, y : \mathsf{Ob}, f, g : \mathsf{Hom}(x, y) \vdash \mathsf{EqHom}(f, g) \\
&x : \mathsf{Ob} \vdash \mathsf{id} : \mathsf{Hom}(x, x) \\
&f : \mathsf{Hom}(x, y) \vdash (\mathsf{id}(x) \circ f) = f : \mathsf{Hom}(x, y) \\
&\ldots
\end{aligned}
$$

- $\mathbb{T}_s$ is an equational extension of $\mathbb{T}_w$.

$$
\mathbf{Alg}(\mathbb{T}_w) \underset{R}{\overset{L}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathbf{Alg}(\mathbb{T}_s)
$$

- $\mathbf{Alg}(\mathbb{T}_w)$ and $\mathbf{Alg}(\mathbb{T}_s)$ have a homotopy theory (weak model structures).

  Sorts $(\Gamma \vdash A)$ correspond to cofibrations $\mathbf{0}_{\mathbb{T}_s}[\Gamma] \rightarrowtail \mathbf{0}_{\mathbb{T}_s}[\Gamma.A]$.

$$\{\} \rightarrowtail \{x\}$$
$$\{x, y\} \rightarrowtail \{x \to y\}$$
$$\{x \rightrightarrows y\} \rightarrowtail \{x \to y\}$$

  Sorts that are sent to trivial cofibrations are called trivially fibrant sorts.

$$(x : \mathsf{Ob} \vdash (y : \mathsf{Ob}) \times (e : x \cong y))$$
$$(x, y : \mathsf{Ob}, f : \mathsf{Hom}(x, y) \vdash (g : \mathsf{Hom}(x, y)) \times \mathsf{EqHom}(f, g))$$

- Their homotopies theories are compatible (Quillen adjunction).

> We want to prove that we have a Quillen equivalence:
> for every cofibrant $\mathcal{M} \in \mathbf{Alg}(\mathbb{T}_w)$, the unit $\eta_{\mathcal{M}} : \mathcal{M} \to R(L(\mathcal{M}))$ is a weak equivalence.

Setup: equational extension

What is an equational extension?

The new equations should correspond to homotopies in the weak theory.

A set[1] H of pairs of points of trivially fibrant sorts.

$$\Gamma \vdash x, y : A \qquad \text{for some trivially fibrant sort } A$$

Typically pairs of (y, h) and (x, r_x) in $(z : A) \times P_A(x, y)$ for homotopies

$$\Gamma \vdash h : P_A(x, y).$$

The strict theory $\mathbb{T}_s$ has rules $(\Gamma \vdash x = y)$ for every pair $(x, y) \in H$.

Monoidal categories:

$$\{(x, y, z : \mathsf{Ob} \vdash \alpha_{x,y,z} : x \otimes (y \otimes z) \cong (x \otimes y) \otimes z),$$
$$(x : \mathsf{Ob} \vdash \lambda_x : x \otimes \mathbf{1} \cong x),$$
$$(x : \mathsf{Ob} \vdash \rho_x : \mathbf{1} \otimes x \cong x)\}$$

[1] or more generally a category: specify how later equations depend on previous equations.

Coherent homotopies

We want to identify which homotopies should be used to transport in strictification.

Arbitrary homotopies = ∞-groupoids (objects, isomorphisms, equalities), (terms, identifications, ...), ...
Coherent homotopies = Other ∞-groupoid structures on objects/morphisms/types/terms/...

What is required from "coherent homotopies":

- Closed under ∞-groupoid operations (including transport in dependent sorts).
- Respect all operations of the algebraic structure.
- Include 1-cells $(x \sim y)$ for every $(\Gamma \vdash x, y : A) \in H$.
- The ∞-groupoids are actually 0-truncated.

Higher algebras

We need a notion of higher algebra of $\mathbb{T}_w/\mathbb{T}_s$:
algebras which have ∞-groupoids as components.

Multiple possible approaches: e.g. one can consider ∞-theories with ∞-categories of algebras.

Our approach: cubical algebras, i.e. algebras internal to cubical sets.
Cubical monoidal categories have a cubical set of objects, a cubical family of morphisms, ...
When this cubical families are fibrant, we can see them as ∞-groupoids.

Cubical algebras

Example of cubical algebras:

- Discrete cubical algebras $\Delta : \mathbf{Alg}(\mathbb{T}_w) \to \mathbf{cAlg}(\mathbb{T}_w)$.
  Cubical paths are trivial.
  There is a right adjoint $1_\square^* : \mathbf{cAlg}(\mathbb{T}_w) \to \mathbf{Alg}(\mathbb{T}_w)$.
- Strict Rezk completions[2] $I : \Delta\mathcal{M} \to \mathsf{SRC}(\Delta\mathcal{M})$.
  Cubical paths coincide with homotopies in $\mathcal{M}$.
  - Components of $\mathsf{SRC}(\mathcal{M})$ are fibrant.
  - Components of $\mathsf{SRC}(\mathcal{M})$ at trivially fibrant sorts are trivially fibrant.
  - $1_\square^*(I) : \mathcal{M} \overset{\sim}{\rightarrowtail} 1_\square^*(\mathsf{SRC}(\Delta\mathcal{M}))$ is a trivial cofibration.

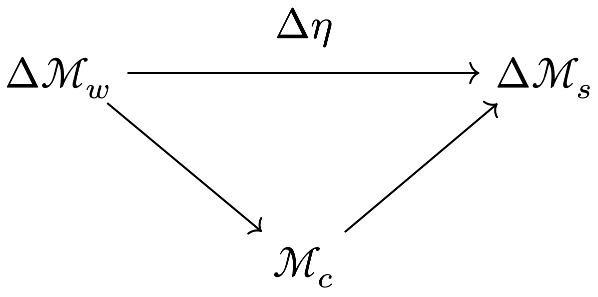
$$\begin{array}{ccc} \Delta\mathcal{M}_w & \xrightarrow{\ \eta\ } & \Delta\mathcal{M}_s \\ \downarrow & & \downarrow \\ \mathsf{SRC}_w(\Delta\mathcal{M}_w) & \longrightarrow & \mathsf{SRC}_s(\Delta\mathcal{M}_s) \end{array}$$

[2] They don't always exist.

The cubical algebra $\mathcal{M}_c$

For a given cofibrant algebra $\mathcal{M}_w$, construct the cubical algebra $\mathcal{M}_c$ freely generated by:

- A morphism $\Delta\mathcal{M}_w \to \mathcal{M}_c$.
- Composition structures at every sort.
- For every pair $(\Gamma \vdash x, y : A) \in H$ and $\gamma : (\mathcal{M}_c)_\Gamma$, a cubical path $x(\gamma) \sim y(\gamma)$.



- In $\mathcal{M}_w$, the identifications for pairs in H are weak.
- In $\mathcal{M}_s$, the identifications for pairs in H are strict.
- In $\mathcal{M}_c$, the identifications for pairs in H are cubical.

Example: monoidal categories

$\mathcal{M}_c$ is the cubical monoidal category generated by:

- A cubical monoidal functor $\Delta\mathcal{M}_w \to \mathcal{M}_c$.
- Composition structures on objects, morphisms (over pairs of objects) and equality of morphisms (over parallel morphisms).
- For every $x, y, z \in \mathcal{M}_c$, a path $\tilde{\alpha}$ between $\big(x \otimes (y \otimes z), \alpha_{x,y,z}\big)$ and $((x \otimes y) \otimes z, \mathrm{id})$.
- For every $x \in \mathcal{M}_c$, a path $\tilde{\lambda}$ between $(x \otimes \mathbf{1}, \lambda_x)$ and (x, id).
- For every $x \in \mathcal{M}_c$, a path $\tilde{\rho}$ between $(\mathbf{1} \otimes x, \rho_x)$ and (x, id).

Let $\mathcal{M}_w$ be a cofibrant $\mathbb{T}_w$-algebra such that:

1. $\Delta\mathcal{M}_w$ has a strict Rezk completion in $\mathbf{cAlg}(\mathbb{T}_w)$.
2. $\Delta\mathcal{M}_s$ has a strict Rezk completion in $\mathbf{cAlg}(\mathbb{T}_s)$.[3]
3. The cubical $\mathbb{T}_w$-algebra $\mathcal{M}_c$ has 0-truncated components.

Then $\eta_{\mathcal{M}} : \mathcal{M}_w \to \mathcal{M}_s$ is a weak equivalence.

[3] I would hope that the following suffices: a cofibrant replacement of $\Delta\mathcal{M}_s$ in $\mathbf{cAlg}(\mathbb{T}_w)$ has a strict Rezk completion.

Application: monoidal categories

Claim: if $\mathcal{M}_w$ is a cofibrant category, the components of $\mathcal{M}_c$ are 0-truncated.

- Immediate for morphisms and equalities between morphisms.
- By a normalization argument[4] $\mathcal{M}_c$.Ob is a retract of the h-set List$(\mathsf{Ob}(\Delta\mathcal{M}_w))$.

  There is a map $\mathsf{fold}_\otimes : \mathsf{List}(\mathsf{Ob}(\Delta\mathcal{M}_w)) \to \mathcal{M}_c.\mathsf{Ob}$.

  Some lemmas are proven by induction on lists:

  1. $p(l_x, l_y) : \mathsf{fold}_\otimes(l_x \mathbin{+\!\!+} l_y) \sim \mathsf{fold}_\otimes(l_x) \otimes \mathsf{fold}_\otimes(l_y)$.
     Needed to interpret $(x \otimes y)$.
  2. Equality between two paths from $\mathsf{fold}_\otimes(l_x \mathbin{+\!\!+} l_y \mathbin{+\!\!+} l_z)$ to $\mathsf{fold}_\otimes(l_x) \otimes \mathsf{fold}_\otimes(l_y) \otimes \mathsf{fold}_\otimes(l_z)$.
     Needed to interpret $\tilde{\alpha}_{x,y,z}$.
  3. Some other lemmas for $\tilde{\lambda}$ and $\tilde{\rho}$.

[4] Similar to "Extracting a Proof of Coherence for Monoidal Categories From a Formal Proof of Normalization for Monoids" [Beylin and Dybjer 1995]

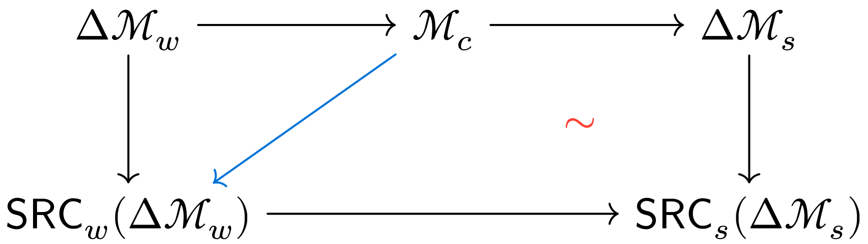
Planned application: type theories

For type theory, prove that $\mathbf{0}_c$ is 0-truncated from "homotopy normalization".

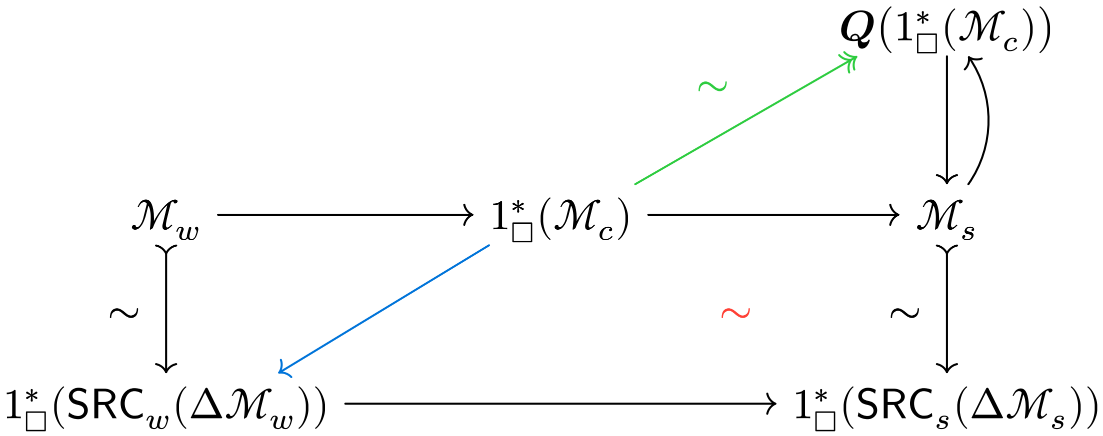
- Similar to "Normalization and coherence for ∞-type theories" [Uemura 2022].
- Strict normalization: sets of terms are retracts of decidable sets of normal forms.
- Homotopy normalization: spaces of terms are retracts of h-sets of normal forms.
- Replace sets by fibrant cubical sets in an algebraic normalization proof.
- Problem: the composition structures of $\mathbf{0}_c$ are not stable under substitution or renamings. Normalization proofs involve presheaves over renamings.
- Two potential approaches:
  1. Generalize strict Reck completions to stable composition and extension structures...
  2. Strictify the composition structures of $\mathbf{0}_c$: use the minimal set of variables supporting a term.

Diagrammatic proof sketch for the main result

In cubical sets:



Externally:



Conclusion

- General reduction of strictification to coherence.
  1. **1.** Strictification: for every cofibrant weak algebra $\mathcal{M}_w$, the unit $\eta : \mathcal{M}_w \to \mathcal{M}_s$ is a weak equivalence.
  2. **2.** Coherence: the components of $\mathcal{M}_c$ are 0-truncated (as families of cubical sets).
- At least sufficient for monoidal categories.
- Conservativity of HoTT over Axiomatic HoTT and other similar results seem within reach of this method.

- The result is constructive but $\eta : \mathcal{M}_w \to \mathcal{M}_s$ is a non-split weak equivalence.
  - It should be possible to get a split weak equivalence when $\mathbb{T}_s$ is well-behaved?
- Possible generalization: avoid strict Rezk completions in $\mathbf{cAlg}(\mathbb{T}_s)$?
- Can we prove 0-truncatedness without normalization / in a weak metatheory?