Towards an Agda formalisation of directed univalence in triangulated type theory

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Simplicial type theory, introduced originally by Riehl and Shulman [10], augments Homotopy Type Theory with a directed interval. This allows us to define synthetic morphisms in types as well as isolate types which have a coherent notion of composition – those which correspond to Segal spaces in the standard model in simplicial spaces. In recent years there have been a number of works that develop the synthetic theory of ∞-categories inside of simplicial type theory ([10],[1],[2],[7]).

In its original incarnation, simplicial type theory consisted of a strict shape layer, where equations on the interval hold definitionally, and strict extension types - similar to the path types of cubical type theory. A proof assistant (Rzk [9]) was developed that implemented this bespoke type theory, and there was a community effort to formalise foundational results – leading up to a proof of the Yoneda lemma in Rzk [5]. Later work in simplicial type theory forgoes this bespoke type theory and instead chooses to work in HoTT with an axiomatised interval. Generalising further, we may choose not to require that our interval be totally ordered, leading to a theory called triangulated type theory [3]. This is the setting of our formalisation and library, which originally set out with two goals:

- To show that the extra coherence obligations that arise as a result of not having a strict interval are not prohibitive for formalisation.
- To take advantage of the modal features of Agda so that we can construct non-trivial examples of categories.

To this end, we present a new Agda library: agda-synthetic-categories, and discuss the design decisions and strategies that allow us to achieve the stated goals. Arguments translated directly from the original work on simplicial type theory or from the Rzk proof assistant often become extremely unwieldy without the strict shape layer; however, we have found that they can often be substituted with higher level arguments which can often provide additional insight – the theory of orthogonality stands out as a particularly useful tool to avoid some of the coherence issues that can otherwise arrise. This library contains formalisations of key results in the foundations of simplicial type theory, such as the (dependent) Yoneda lemma [13], as well as an ongoing project on the construction of the directed univalent category of spaces. As part of this project, an extension to the modality system in Agda has been developed and upstreamed, implementing the $\sharp$ modality of cohesive type theory ([12], [8]).

The most technical part of this formalisation pertains to the construction of the category of spaces, and we highlight some of the key results and definitions needed for the formalisation. The construction is related to the internal universes considered by Licata, Orton, Pitts, and Spitters [6], and crucially makes use of a right adjoint to exponentiating by the interval, a so-called *amazing right adjoint.*

Definition 1 ([3, Axiom 3]) *We say that a type I is tiny if for each crisp type A, there is a type A_I and a map $\varepsilon_A : (A_I)^I \to A$, such that for any crisp type Γ, the induced map $\flat(\Gamma \to A_I) \to \flat(\Gamma^I \to A)$ is an equivalence. The operator $-_I$ is called the 'amazing right adjoint'.*

Unlike approaches which natively include a 'tiny type', such as Riley's Tiny Type Theory [11] or the multi-modal approach taken in earlier versions of Gratzer, Weinberger, and Buchholtz's work on directed univalence, we are able to prove internally that tiny types are preserved under products and retracts, and that any tiny type is projective. Following Gratzer, Weinberger, and Buchholtz [3], the formalisation derives a dependent version of the amazing right adjoint, from which we give a general construction for 'amazing' properties – the paradigmatic example being amazing covariance as introduced in op. cit.

Definition 2 *Fixing a type I, a class of maps $\mathcal{F}$ is called I-*relative *if for all maps $f : A \to B$, $f \in \mathcal{F}$ iff for each map $\rho : I \to B$, the pullback $\rho^* f$ belongs to $\mathcal{F}$.*

Theorem 3 *Given a tiny type I, and a subclass of maps $\mathcal{F}$ which is I-relative, we can construct a type U_F of amazingly fibrant types, and a map $U^\bullet_{\mathcal{F}} \to U_{\mathcal{F}}$ in $\mathcal{F}$, which classifies crisp small maps in $\mathcal{F}$.*

Immediately we obtain $(\mathcal{S}^\bullet \to \mathcal{S})$, the classifier for left fibrations, but also $\mathcal{S}^{op}$, the classifier for right-fibrations.[1] The formalisation aims to prove the following results about $\mathcal{S}$:

Theorem 4 *$\mathcal{S}$ is a directed univalent category:*

1. *As a type it is simplicial, Segal and Rezk complete*
2. *There is an equivalence* $\mathrm{Hom}(A, B) \simeq (A \to B)$ *for each A $B : \mathcal{S}$*

Currently only the second part of this theorem is formalised, assuming axioms 1-8 of [3], as well as a pair of 'redundant' axioms, one of which follows from duality and the theory of cofinal maps, and the other is a small technical lemma on tiny types. In future work, in addition to completing the proof of theorem 4, we hope to extend this formalisation to construct the category of ∞-categories ([4]) and the category of profunctors, which has not yet been considered in the literature. We expect the framework presented in this talk to generalise to this setting; however, a crucial prerequisite, the theory of (co)cartesian fibrations (or exponential fibrations) has yet to be formalised in the library.

[1] We note that unlike in [3] we do not assume that opposite categories exist in general, either through modalities or additional axioms.

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