A Path-Type Weak Factorization System

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Our main contribution is to construct a weak factorization system (wfs) in the recently developed semantics of intensional identity types by Awodey and Hua [4] based on path types. This wfs is functorial and cloven, and covers the groupoid model as an example.

The path types semantics came out of the HoTTLean project [13], where it was realized in the Lean formalization of the groupoid model that the semantics of intensional identity types can be simplified by using an interval.

Following Awodey and Hua, we work in a finite limit category $\mathcal{E}$ with a distinguished map $\mathsf{tp} : \mathsf{Tm} \to \mathsf{Ty}$ called a *universe*. The universe *classifies* a class $\mathcal{D}$ of *type families*, comprising all pullbacks of tp. In addition, we suppose $\mathcal{E}$ contains an *interval*, which is an exponentiable, bipointed object $d_0, d_1 : 1 \rightrightarrows I$, regarded as an interval with two endpoint inclusions.

We assume the universe satisfies the following axioms of Awodey and Hua.

Axiom 1 ([4, A1])**.** The universe tp is a normal Hurewicz fibration.

Recall that a map $p : E \to B$ is a *Hurewicz fibration* when, for each object X in $\mathcal{E}$, p has the right lifting property[1] against $X \times d_0$; i.e. $X \times d_0 \pitchfork p$ as depicted below.

$$\begin{array}{ccc} X & \longrightarrow & E \\ {\scriptstyle X\times d_0}\downarrow & \nearrow & \downarrow{\scriptstyle p} \\ X \times \mathbf{I} & \longrightarrow & B \end{array}$$

Equivalently, there is a section ℓ of the pullback-hom[2] $d_0 \Rightarrow p : E^{\mathbf{I}} \to B^{\mathbf{I}} \times_B E$, providing a solution to the generic lifting problem. Call ℓ the *path lifting operation* of p. A Hurewicz fibration is *normal* when ℓ lifts reflexivity paths to reflexivity paths; formally, denoting the reflexivity inclusion $\rho_X = X^{!_{\mathbf{I}}} : X \to X^{\mathbf{I}}$, normality says that $\rho_E = \ell \langle \rho_B p, 1_E \rangle$. As pullbacks of the universe tp, type families inherit normal Hurewicz fibration structure.

Axiom 2 ([4, A2])**.** The universe tp *has path types*, in the sense that there are maps $(\mathsf{Path}, \mathsf{path})$ making the diagram

$$\begin{array}{ccc} \mathsf{Tm}^{\mathbf{I}} & \xrightarrow{\mathsf{path}} & \mathsf{Tm} \\ {\scriptstyle \mathsf{tp}^{[d_0,d_1]}}\downarrow & \lrcorner & \downarrow{\scriptstyle \mathsf{tp}} \\ \mathsf{Tm} \times_{\mathsf{Ty}} \mathsf{Tm} & \xrightarrow[\mathsf{Path}]{} & \mathsf{Ty} \end{array}$$

a pullback square, where the exponential by $\mathbf{I}$ and its boundary projections are taken in the slice over Ty.

Awodey and Hua [4] prove that the axioms above soundly model intensional identity types with coherent substitution. That is, the type families in $\mathcal{D}$ satisfy the formation, introduction, elimination, and computation rules of identity types. Formation and introduction arise respectively from the structure maps Path and path.

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[1] See [9, §4] for the definition of $f \pitchfork g$, $f \pitchfork \mathcal{R}$, ${}^{\pitchfork}\mathcal{R}$, etc.
[2] Refer to [12].

Elimination (so-called J-rule) and computation correspond to lifting problems of reflexivity inclusions ρ_X against type families $E \to X^{\mathbf{I}}$ over the path type.[3] They use the path types and fibration structure to directly construct solutions to such lifting problems [5, Proposition 2.7]. Our wfs given below subsumes this result by encompassing reflexivity inclusions ρ_X as left maps and type families $E \to X^{\mathbf{I}}$ as right maps.

To the axioms above of Awodey and Hua, we add just one:

Axiom 3. The universe tp *has* Σ *types.*

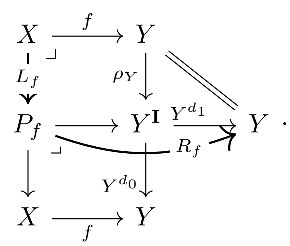
This is to be formulated in terms of polynomial composition, as in [4, 3]. It then follows that $\mathcal{D}$ is closed under composition.

Under these three axioms, we obtain a wfs analogous to Gambino and Garner's *identity type weak factorization system* [9]. However, by working in a setting with a universe and functorial path types afforded by the interval, we obtain greater coherence than possible in the bare syntactic category. Thus, our main contribution is a proof of the following theorem.

Theorem. *For a universe* tp *satisfying Axioms 1–3 in a finite-limit category, let* $\mathcal{C}$ *denote the full subcategory spanned by the objects* X *whose map to* 1 *is in* $\mathcal{D}$*. Let* $\mathcal{L}$ *be the restriction of* ${}^{\pitchfork}\mathcal{D}$ *to* $\mathcal{C}$ *and* $\mathcal{R}$ *the restriction of* $\mathcal{L}^{\pitchfork}$*. Then the pair* $(\mathcal{L}, \mathcal{R})$ *forms a wfs on* $\mathcal{C}$ *which*

1. *Coincides with (strong deformation retracts*[4]*, normal Hurewicz fibrations).*
2. *Has a functorial* path object factorization.
3. *Admits an explicit* cleavage *induced by the universal structure on* tp.

For the proof, we factorize each map $f : X \to Y$ using the standard *path object factorization* $f = R_f L_f$ as in the diagram



(1)

This factorization inherits functoriality from that of $(-)^{\mathbf{I}}$.

Essentially, the whole wfs structure is induced by the universe tp, in the following sense. Awodey and Hua show that the universe tp admits a *connection* $\chi : \mathsf{Tm}^{\mathbf{I}} \to (\mathsf{Tm}^{\mathbf{I}})^{\mathbf{I}}$, which assigns to each path $p : a \rightsquigarrow b$ in Tm a higher path from the reflexivity path at a to p itself [4, Lemma 2.6]. Each type inherits a connection from the universal one, which in turn gives rise to a strong deformation retraction of any left factor L_f. Each right map inherits a normal path lifting ℓ from the universal one. In terms of this structure, we are able to give an explicit *cleavage* of the wfs, which is a structure witnessing a *cloven* wfs, introduced in [6, Definition 3.2.1] as a middle-ground between structureless and fully algebraic [10] wfs. This approach—imposing structure on a universe, which is then inherited by the things it classifies—is inspired by algebraic set theory and has been taken up more recently for type theory in [2].

Example (Groupoids)**.** Hofmann and Streicher's groupoid model [11] provides a first example. The universe $\mathsf{tp} : \mathsf{core}(\mathsf{PGrpd}) \to \mathsf{core}(\mathsf{Grpd})$ is the forgetful functor of (the core of) small pointed groupoids, sending a pointed groupoid $(G, g \in G)$ to G. The interval $\mathbf{I}$ is the walking isomorphism, making the path type $G^{\mathbf{I}}$ of a groupoid G its arrow groupoid. Then the associated wfs is (injective adjoint equivalences, small cloven isofibrations) restricted to small groupoids. For a functor $f : G \to H$ of groupoids, its path object P_f is the isocomma groupoid $f \downarrow_{\cong} H$ comprising pairs $(a, p : f(a) \to b)$ where a is an object in G and p is an

[3]See [5] for details.
[4]See any text on homotopy theory, e.g. [7, Definition 2.4.16].

isomorphism in H. Now the left factor $L_f : G \to P_f$ sends $a \mapsto (a, 1_{f(a)})$ and the right factor $R_f : P_f \to G$ projects $(a, p : f(a) \to b) \mapsto b$. The connection $\chi : G^{\mathbf{I}} \to G^{\mathbf{I}\times\mathbf{I}}$ of a groupoid G sends an isomorphism $p : a \to a'$ in G to the square $(1_a, p) : 1_a \to p$. See [9, §5] for a comparison to the identity type wfs.

The idea of homotopy theoretic models of dependent type theory goes back to Awodey and Warren [5], inspired by the groupoid model [11]. They show that every wfs *soundly* models (non-coherent) identity types. Gambino and Garner show *completeness* by building a wfs on the syntactic category [9]. Obtaining *path types* from factored diagonals comes from [5]. The *path object factorization* described above seems to go back to van den Berg and Garner, who show that it can be carried out in any *path object category* [6]. While they assume an internal category structure on the path object, we do not; and while we assume a universe, they do not. Later, van den Berg and Moerdijk [15] show that the factorization can be performed in more general settings. Although they do not obtain a full wfs (lifting problems there admit solutions only "up to homotopy"), their left maps coincide with ours as strong deformation retracts, as do the left maps in [6] and [9]. Emmeneger [8] gives a general construction of a wfs similar to ours, in the presence of a *clan* and *stable path object* (exponentiation by an interval is an instance). Awodey uses a bare bipointed interval for path object factorizations in [1, 3]. However, the *biased fibrations* there lift against *both* endpoints and the *unbiased fibrations* there lift against the "generic point", giving both stronger lifting properties than the Hurewicz fibrations in our work. Meanwhile, we assume *normality* which Awodey [3] does not. Thus, neither our wfs nor Awodey's subsumes the other. In future work, we hope to further investigate the connection between our work and these other interval-based semantics of identity types.

We expect to find more examples, such as in simplicial sets. Ultimately, we are interested in how close our minimal setting brings us to a Quillen model structure [14]. One open question is to what degree this framework is "biased" towards the 0-endpoint. Concretely, assuming `tp` lifts against 0-endpoint inclusions as above, does `tp` also lift against 1-endpoint inclusions? Since, as [4] showed, the universe `tp` models identity elimination, it is possible to show that any type family lifts against 1-endpoints *up to homotopy*. We hope to find that this need not hold *strictly*, in an example where the interval is not strictly invertible. Finally, we plan to formalize our work in the Lean proof assistant, continuing from the existing formalization in [4].

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