HoTTLean: Formalizing the groupoid model of MLTT

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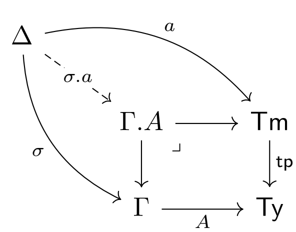
Contributions. In the proof assistant Lean, we formalize the groupoid model of Martin-Löf Type Theory (MLTT) [HS98]. This model admits a finite hierarchy of universes, Σ-types, Π-types, and Id-types. Though these results are well-known on paper, our work makes a novel contribution to the formalization of the groupoid semantics of MLTT, bridging the gap between synthetic and classical mathematics. This work is part of the broader HoTTLean project.

The groupoid model. In HoTTLean, rather than constructing an ad hoc interpretation of MLTT syntax in the groupoid model, we construct an interpretation into a generic model of type theory (not the focus of this paper) – called an "elementary model" in [HX26] – and verify that the groupoid model satisfies the axioms for an elementary model. Though the groupoid model construction itself is not particularly modular, the interpretation of syntax into an elementary model can be reused for other models, as detailed in [NHC⁺26].

The formalization repository can be found at github.com/sinhp/HoTTLean. In the following, we also include links to the formalized counterparts of definitions, labeled using the symbol ◫.

Elementary models. An elementary model is roughly equivalent to a series of nested natural models with Π-types, Σ-types and "large eliminating" Id-types, which are all small (i.e. in the image of the Yoneda embedding). A brief overview follows.

By a *universe* in a category $\mathcal{C}$, we will mean a morphism $\mathsf{tp} : \mathsf{Tm} \to \mathsf{Ty}$ that has pullbacks along all maps $A : \Gamma \to \mathsf{Ty}$. We denote the pullback of tp along A using the context extension notation $\Gamma.A$. We denote the substitution determined by $\sigma : \Delta \to \Gamma$ and $a : \Delta \to \mathsf{Tm}$ such that $a \gg \mathsf{tp} = \sigma \gg A$ using the notation $\sigma.a : \Delta \to \Gamma.A$.



A elementary Σ-type structure on tp consists of the following. For any context Γ, type $A : \Gamma \to \mathsf{Ty}$, and dependent type $B : \Gamma.A \to \mathsf{Ty}$ there is a type $\Sigma_A B : \Gamma \to \mathsf{Ty}$. For any

$a : \Gamma \to \mathsf{Tm}$ and $b : \Gamma \to \mathsf{Tm}$ such that $a \gg \mathsf{tp} = A$ and $b \gg \mathsf{tp} = \mathsf{id}_\Gamma.a \gg B$, there is a map $\mathsf{pair}(a,b) : \Gamma \to \mathsf{Tm}$ satisfying $\mathsf{pair}(a,b) \gg \mathsf{tp} = \Sigma_A B$. For any $s : \Gamma \to \mathsf{Tm}$ such that $s \gg \mathsf{tp} = \Sigma_A B$, there are maps $\mathsf{fst}\, s : \Gamma \to \mathsf{Tm}$ and $\mathsf{snd}\, s : \Gamma \to \mathsf{Tm}$ satisfying $\mathsf{fst}\ s \gg \mathsf{tp} = A$ and $\mathsf{snd}\ s \gg \mathsf{tp} = (\mathsf{id}_\Gamma.\mathsf{fst}\ s) \gg B$. All of these constructions are stable under substitutions $\sigma : \Delta \to \Gamma$, and the following computation rules hold:

$$\mathsf{fst}\ (\mathsf{pair}(a,b)) = a \qquad \mathsf{snd}\ (\mathsf{pair}(a,b)) = b$$

$$\mathsf{pair}(\mathsf{fst}\ s, \mathsf{snd}\ s) = s$$

We refer to [HX26] for the definition of universe lifts, elementary Π-types and Id-types. An elementary model consists of a category $\mathcal{C}$, with a sequence of universe lifts $\mathsf{tp}_0 \to \cdots \to \mathsf{tp}_n$, such that each universe has the structure of Π-types, Σ-types, and Id-types that eliminate into all other universes.

Contexts ⧉. The category interpreting contexts is the category of groupoids, denoted Ctx. In Mathlib [The20], there are two universe level variables associated with the category of groupoids, one for the size of objects and one for the size of hom-sets, which we both take to be 4 (arbitrarily).

$$\mathsf{Ctx} := \mathsf{Grpd}.\{4,4\}$$

Due to the constraints of universe postulates in Lean, we do not take variables as the universe levels for the category of groupoids; doing so would prevent us from constructing universes.

Universes ⧉. We provide a universe for each Lean universe-level $u < 4$ polymorphically: Ty is the core of the category of (u-small) groupoids, Tm is the core of the category of (u-small) pointed groupoids, $\mathsf{tp} : \mathsf{Tm} \to \mathsf{Ty}$ is the forgetful functor, and pullbacks of tp are computed as Grothendieck constructions.

Universe lifts ⧉. Universes in the model are related by universe lifts $\mathsf{tp}.\{u\} \to \mathsf{tp}.\{u+1\}$, for each $u < 3$. These are constructed using the embeddings $\mathsf{Grpd}.\{u,u\} \to \mathsf{Grpd}.\{u+1,u+1\}$.

Σ-types ⧉. For the formation rule of sigma types, we require for each pair of functors $A : \Gamma \to \mathsf{Ty}$ and $B : \Gamma.A \to \mathsf{Ty}$, a functor $\Sigma_A B : \Gamma \to \mathsf{Ty}$. This amounts to constructing for each pair of functors $A' : \Gamma \to \mathsf{Grpd}$ and $B' : \int A' \to \mathsf{Grpd}$, a functor $\Sigma_{A'} B' : \Gamma \to \mathsf{Grpd}$. This can be done "pointwise" using Grothendieck constructions.

$$\Sigma_{A'} B'(x) = \textstyle\int (B|_{A'x} : A'x \to \int A' \to \mathsf{Grpd})$$

This construction of Σ-types is automatically universe-polymorphic: if A is u-small and B is v-small, then $\Sigma_A B$ is $\max(u,v)$-small.

Π-types ⧉. Our elementary construction of Π-types uses the construction of Σ-types: the formation of Π-types is defined by taking "pointwise sections" of the Σ-types. As a consequence of constructing Π-types this way, it is easier to not construct Π-types polymorphically – if A is u-small and B is u-small, then $\Pi_A B$ is u-small. Then, to obtain universe-polymorphic Π-types, we simply combine the above construction with universe lifts.

Id-types. By the results in [AH26], to construct Id-types in the groupoid model, it suffices to identify an interval structure on the category of groupoids, and to construct path types using the interval. Like before, we work with an elementary formulation of path types, described in [AH26, Section 5]. The "walking isomorphism" I plays the role of the interval, and isofibrations are the Hurewicz fibrations.

Related work. We list some other projects with the goal of formalizing the semantics of MLTT. For a broader account of related work, see [NHC+26]. Du [Du25] formalized in Lean a simplicial model of type theory as a contextual category with Π-types, a single universe, and no univalence axiom, modulo a Kan-Quillen model structure on simplicial sets. Sozeau and Tabareau [ST14] worked towards a Rocq formalization of the groupoid model, but this has not been completed, to our knowledge. The groupoid model is formalized as an example comprehension category in the Unimath project [GAM+25], also without type formers.

Future work: HoTT0. We have not shown that the groupoid model of MLTT is a model of HoTT0, a fragment of HoTT in which univalence holds only on the set-truncated types [HAC+25]. Though a brute-force proof of the axioms should be fairly straight-forward, we would prefer to first create a general user interface for this kind of syntax-semantics reasoning, using the tools developed in SynthLean [NHC+26].

We have not constructed $0, 1$ or W-types in the groupoid model, nor have we defined them in terms of the elementary model.

Future work: algebraic models. Constructing an alternative "algebraic" groupoid model – described in [HX26, Example 5.14] – and comparing it with the elementary groupoid model would provide general insight for future formalizations of MLTT models.

There are limitations to constructing the groupoid model as an elementary model. The constructions of type formers in an elementary model leads to the appearance of more type equalities, which in Lean is generally undesirable. The elementary approach also requires explicit proofs of stability of all operations under substitution whereas an algebraic one can avoid such considerations by using a universal construction. Though these proofs about substitution are usually straightforward, moving this proof burden elsewhere makes the algebraic construction of the groupoid model more modular; stability under substitution is worked out once-and-for-all when converting a general algebraic model to an elementary one [HX26].

Our elementary approach does not require identifying categorical structure on Grpd. For example, we did not prove that split isofibrations are closed under pushforwards[1] or that the interval I is exponentiable. Not having to prove this may be considered beneficial, though such a structure in Grpd is of interest in and of itself and would be a meaningful contribution to Mathlib.

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