# Towards Formalization of Directed Univalence in Rzk proof assistant

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**Background and Motivation.** Simplicial Type Theory (STT) by Riehl and Shulman serves as a foundation for synthetic $\infty$-category theory [RS17]. It extends Martin-Löf Type Theory (MLTT) with a notion of directed interval, enabling synthetic reasoning not only about $\infty$-groupoids as in classical homotopy type theory, but about $\infty$-categories.

Triangulated Type Theory ($\triangle$TT), introduced by Gratzer, Weinberger, and Buchholtz [GWB24], enriches simplicial type theory with three modalities $\flat$, $\sharp$, op. More concretely, these modalities can be described as follows:

- $\flat$ – the discrete modality, removes higher cells and leaves only a groupoid of points;
- $\sharp$ – the codiscrete modality, the right adjoint to $\flat$;
- op – the opposite modality, reverses orientation of arrows.

$\triangle$TT enables the construction of an internal category of groupoidal types, along with a proof of directed univalence for that universe. Additionally, Gratzer et al. provide a construction for a number of large categories, such as the category of monoids and the category of reflexive graphs. Further results in this direction include the Yoneda embedding and the category of $\infty$-categories [GWB26; GWB25].

Rzk [Kud+26b] is an experimental proof assistant for simplicial type theory, featuring a three-layer architecture of cubes, topes, and types, as well as extension types (the implementation follows STT [RS17]). The coherences in the tope layer are checked via a dedicated intuitionistic constraint solver, allowing more direct reasoning about shapes in proofs. There already exist some formalisation projects built using Rzk [KRW24; Kud+26c; Kud+26a]. However, until now Rzk did not support modal reasoning, preventing the formalization of Triangulated Type Theory.

**Contribution.** We formalize Triangulated Type Theory in Rzk, building on an extension of the proof assistant with modal reasoning. More precisely, we equip the Rzk typechecker itself with a general mode-parametric framework for modalities [Gra+21], which we then instantiate to obtain the modalities of $\triangle$TT. This generality is important for future work: further results in synthetic $\infty$-category theory may require additional modalities — such as a twisted modality — which can be incorporated by extending the typechecker implementation with minimal effort. The experimental version of Rzk with modal reasoning is available in the branch `lishy2-modal` of the Rzk repository `github.com/rzk-lang/rzk`.

We also formalize parts of △TT in Rzk, postulating some of the axioms, formalizing the "amazing covariance" predicate (`is-a-cov`), constructing the universe of groupoid types $\mathcal{S}$ (`S`), establishing the maps `mor2fun` and `dirglue`, which together yield directed univalence. The formalization is available in the branch `diruniv` of the fork of the sHoTT repository `github.com/LIshy2/sHoTT`, in the `triangulated` directory (for the current formalization progress, see § *Current status*).

**Modalities and Topes.** △TT is originally formulated over MTT extended with an interval *type* $\mathbb{I}$, but we want to continue following Riehl and Shulman's extension types and use the judgmental interval *cube* $\mathbb{2}$ from Rzk to keep definitional equalities, making proofs easier. Hence, axiomatization of △TT in Rzk requires adapting to the three-layer setting. Added modalities in Rzk can affect variables in all layers, cubes, and types, but the corresponding terms can only occur in the last (MLTT) layer.

Of the 10 axioms postulated in △TT [GWB24], two axioms (2 and 7) concern the interaction of modalities with the interval structure and must be built into the typechecker, which simplifies proofs by providing judgmental equalities.

- (Axiom 2) *Opposite of $I$*: the op modality reverses the directions in shapes, for which we introduce dedicated operators flip/unflip (for cubes), and inv/uninv (for topes).
- (Axiom 7) *Global points of $I$*: discreteness of the $\flat$-interval is encoded as a tope-level axiom: for every $i : \flat I$, either $i = 0$ or $i = 1$:

  $$i : \flat\mathbb{2} \mid \cdot \vdash (i \equiv 0_{\mathbb{2}}) \vee (i \equiv 1_{\mathbb{2}})$$

  This enables an elimination principle for $\flat I$ similar to that of Bool.

The remaining eight axioms can be postulated directly inside the type theory, but their statements require attention to accommodate the STT setting.

The following comparison table summarizes and highlights some of the nuances between △TT and its adapted version in Rzk:

| | △TT | Rzk |
|---|---|---|
| Directed interval | postulated as type | definitional cube |
| Axiom 2 (Opposite interval) | postulated | built-in flip/unflip, inv/uninv |
| Axiom 7 (Global points) | postulated ($\mathbb{I} \simeq$ Bool) | built-in rule in tope solver |
| Other axioms | postulated | postulated in formalization |
| Univalent universe of groupoids $\mathcal{S}$ | [GWB24, Definition 6.1] | `S` in formalization |

**Current status.** The formalization is still in progress; we hope to have a complete proof of directed univalence (i.e., that `mor2fun` and `dirglue` form an equivalence) by the time of the workshop. We also allow some shortcuts in the formalization, such as the questionable choice of retaining the total order on the interval.

**Talk overview.** The talk will briefly introduce the Rzk proof assistant, focus on the details of the formalization of △TT and directed univalence, and discuss further work in mechanization of more advanced results.

# References


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