ORIENTED SIMPLICES AND CUBES ARE PARAMETRIC

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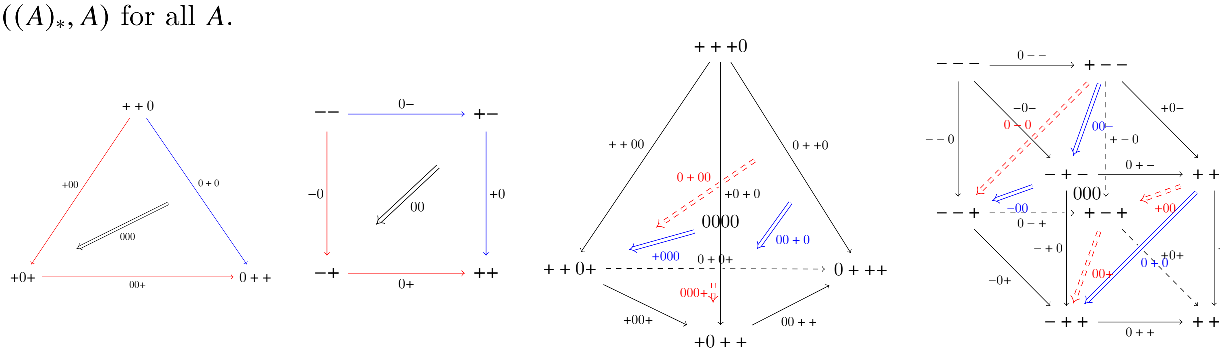
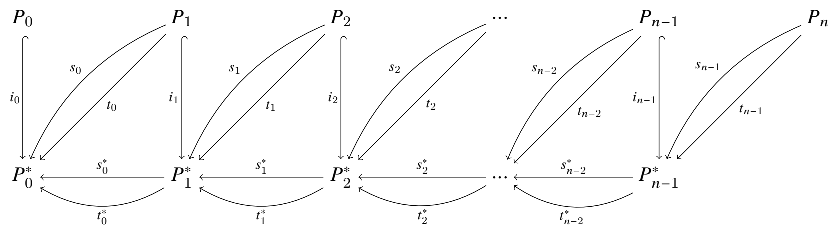
Abstract. Street's Orientals [Str87] construction provides a structure of n-(strict) category to the n-simplex Δ_n via the functor $O : \Delta \to \omega Cat$. Recent work by Ara, Lafont and Métayer [AML23] show how O_n can be seen as a free category on a polygraph of which the generators can be inductively generated. Likewise, we propose a similar construction of oriented cubes and analyze their relation to the oriented simplices following insights found in the work of Aitchison [Ait10]. Both the simplicial and cubical case can be regrouped into a common framework, the one of Reynolds' parametricity [Rey72], specifically the iteration of its unary and binary variants, as seen in the work of Herbelin and Ramachandra [HR23]. The main contribution is to offer a unified computable approach with intuitive notation and moreover building well-defined translations between these two variants.
Keywords: Orientals, Oriented Cubes, Semi-Simplicial Sets, Semi-Cubical Sets, Parametricity, Higher Category Theory, Homotopy Type Theory.

1. Parametricity. The construction and analysis of semi-simplicial and semi-cubical types aims to solve the difficulty of addressing the required iterated coherences of higher structures in Homotopy Type Theory and related foundations (paraphrasing [KS25]). Indeed the (augmented semi-)simplex and (semi-)cube categories have been proven useful to the development of type theory just as they are to algebraic topology and (higher) category theory. However, the difficulty on the standard presentation of semi-cubical sets, that is of presheaves over $\square$, comes from the fact that the corresponding data (in its *fibered* version) is a family of sets $X_0, X_1, ...$, a family of 1-dimensional decreasing projections, and crucially then a family of *coherences* such as $x_{0+} * x_- = x_{-0} * x_+$ and so on, which happen to be related to the construction of orientals. We look then at the polygraphic construction of $O : \Delta \to \omega Cat$ and $\mathcal{I} : \square \to \omega Cat$. Indeed the construction of the simplicial orientals as the iteration of a monad on ωCat corresponds to the *indexed* presentation of such kind of presheaves where the information of coherences has been internalized to the dependency of a level of the construction on the lower ones, as seen in [HR23]. This method for an internalization procedure treating uniformally the simplicial and cubical case relies on iterated parametricity, so we treat a categorical definition of it found in [Moe22], that is as an endofunctor $(\_)_*$ with distinguished morphisms in $Hom((A)_*, A)$ for all A.

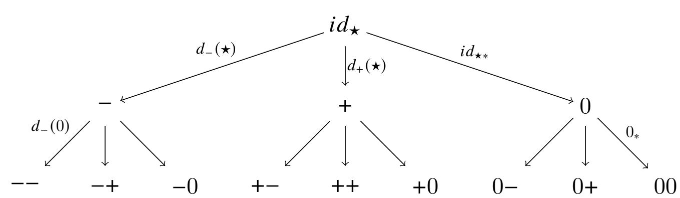


2. Globular v. Simplicial v. Cubical. Street's construction of the simplicial orientals, motivated by cohomology and allowing the construction of the ω-nerve, can in fact be summarized to solving the discrepancy between the *globular* and *simplicial* structures of Δ_n. What is thus required is to provide a coherent description of source and target maps for all faces and an adequate composition such that the simplicial conditions are properly respected. Likewise, the construction of oriented cubes amounts to mimicking this concept with respect to the cubical conditions. Usually 0-cells are denoted by integers $< 0 >$ to $< n >$ in the simplicial case and $[0]$ to $[2^n - 1]$ in the cubical case, so we introduce a notation with uniformity on mind: $+ ... + 0 + ... + := < i >$ if the 0 is on position i from right to left and $\epsilon_n ... \epsilon_0 := [\sum x_i 2^i]$ with $x_i = 0$ if $\epsilon_i = -$ and $x_i = 1$ if $\epsilon_i = +$. The amount of 0's in the notation of a higher cell will denote the dimension (minus 1 in the simplicial case, where the point is $\Delta_0 =: 0$). We use a polygraphic approach to define the orientations.

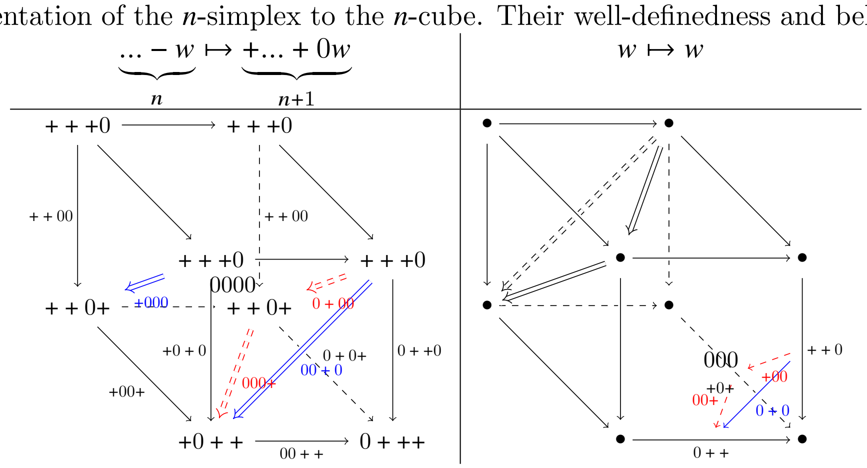
3. Polygraphs and Translations. An n-polygraph is defined as a family of generators P_i, that is i-dimensional cells, inducing free i-categories P_i^* via maps s_i, t_i, stating the source and target of a generator of dimension i in the free $(i-1)$-category P_{i-1}^*:



We refer to [ABG+25] for further details on the definition of polygraphs and its applications. Following the work in [Moe22] we describe $\square$ as the strict monoidal category freely generated from the monoidal unit $\star$, an object 0, and the maps $-, + : 0 \longrightarrow \star$, and Δ_+ as the strict monoidal category freely generated from the monoidal unit $\star$, an object 0, and the map $+ : 0 \longrightarrow \star$. On one hand, this builds a parametric structure on the cube and simplex: $(A)_* := A \otimes 0$, $f_* := f \otimes id_0$ and $d_\epsilon(A) := id \otimes \epsilon$ with $\epsilon \in \{+, -\}$ (just + for the simplex). On the other hand, by this approach the polygraphic i-generators of the n-cube and n-simplex are given by the canonically generated morphisms between $0^{\otimes n}$ and $0^{\otimes i}$. We omit the tensor symbol and replace id_0 by 0, composition thus described by replacing the 0's in order, so that the generators can be organized in generations corresponding to the levels of the following tree (we show the cubical case):



Indeed the generators of level $i \leq (n-1)$ generate the $(n-1)$-cube which injects itself into the n-cube (and likewise for the simplex). This construction also relies on iterated parametricity as the branches correspond to pre-composing $d_\epsilon(A)$ or taking f_*. In the simplicial case, they correspond as well to the shift and expanded shift operations establishing the syntax of orientals in [AML23]. At this point, we can establish the source and target of 0...0 of lenght n in the simplicial case for all n which induces the orientation of all cells of dimension n. We can then define the orientation of the cube as the only orientation that would be compatible with the operations of taking the corner of the cube containing $+...+ := [2^n - 1]$ and of taking its base, which syntactically are a map and a partial map defined for w with no $-$'s defined at the end of the paragraph. As these are orientation preserving identifications of the n-simplex and $(n-1)$-simplex in the n-cube, conversely a polygraphic argument gives that the first one is as well instructions to extend the orientation of the n-simplex to the n-cube. Their well-definedness and behavior are at the heart of the work:



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