AXIOMS FOR HIGHER CATEGORY THEORY

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In a work-in-progress book project [2], Cisinski, Cnossen, Nguyen, and Walde develop higher category theory synthetically from a collection of axioms. Our aim here is to explain how this development can be carried out on top of homotopy type theory, where we interpret types as *spaces*. Specifically, our starting point is homotopy type theory with Σ, $=$, Π, 0, 1, pushouts, $\mathbb{N}$, and a univalent universe $\mathcal{U}$ closed under aforementioned type formers.

To start, we require a "wild" category Cat of *categories*.[1] Informally, an n-wild category (for finite n or $n = \infty$) is the data of a category with coherence only up to level n.[2] This can be encoded in various ways: directly for low values of n or schematically, for example as a Reedy fibrant complete semi-Segal type [1, Section 4.7]. For basic developments, 2-wildness seems to suffice.

Below, we shortcut the infinite coherence tower by asking for functor composition to be judgmentally unital and associative.[3] This is validated by most models of higher categories used in practice and greatly simplifies formalization.

Axiom 1.

1. We have a type Cat of *categories*.
2. For categories C and D, we have a small type $\mathsf{Cat}(C, D) : \mathcal{U}$ of *functors*.
3. Given $C : \mathsf{Cat}$, we have an *identity functor* $\mathsf{id}_C : \mathsf{Cat}(C, C)$.
4. Given $f : \mathsf{Cat}(C, D)$ and $g : \mathsf{Cat}(D, E)$, we have a *composite functor* $g \circ f : \mathsf{Cat}(C, E)$.
5. For $f : \mathsf{Cat}(C, D)$, the terms $f \circ \mathsf{id}_C$, $\mathsf{id}_D \circ f$, and f are judgmentally equal.
6. For composable functors f, g, h, the terms $h \circ (g \circ f)$ and $(h \circ g) \circ f$ are judgmentally equal.

The remaining axioms only postulate terms. These inhabit propositions, with the exception of the enumeration in Axiom 4.

A functor $f : \mathsf{Cat}(C, D)$ is *invertible* if it has a left inverse and a right inverse. We write $C \simeq D$ for the type of invertible functors from C to D.

Axiom 2 (Cat is univalent)**.** For $C : \mathsf{Cat}$, the type $(D : \mathsf{Cat}) \times (C \simeq D)$ is contractible.

Axiom 3. The wild category Cat is cartesian closed.

We denote the exponential of $C, D : \mathsf{Cat}$ by $\mathrm{Fun}(C, D) : \mathsf{Cat}$ with evaluation $\mathrm{ev} : \mathsf{Cat}(\mathrm{Fun}(C, D) \times C, D)$: for $X : \mathsf{Cat}$, the function $\mathsf{Cat}(X, \mathrm{Fun}(C, D)) \to \mathsf{Cat}(X \times C, D)$ given by $f \mapsto \mathrm{ev} \circ (f \times C)$ is an equivalence.

An *object* of $C : \mathsf{Cat}$ is an element of $\mathrm{Ob}(C) := \mathsf{Cat}(1, C)$.

Axiom 4. We have a category $\mathbb{I} : \mathsf{Cat}$ with an equivalence $e : \mathrm{Ob}(\mathbb{I}) \simeq \{0, 1\}$.

We will leave the equivalence e implicit.

A *morphism* of $C : \mathsf{Cat}$ is an element of $\mathsf{Cat}(\mathbb{I}, C)$, with *domain* $\mathsf{dom}(f) := f \circ 0$ and *codomain* $\mathsf{cod}(f) := f \circ 1$. Given $x, y : C$, the *hom-space* $C(x, y)$ is the fiber of $(\mathsf{dom}, \mathsf{cod}) : \mathsf{Cat}(\mathbb{I}, C) \to \mathrm{Ob}(C)^{\{0,1\}}$ over (x, y). Terminality of 1 defines the *identity* $\mathsf{id}_x : C(x, x)$ at $x : C$.

Composition with a functor $f : \mathsf{Cat}(C, D)$ induces actions $f(x) : \mathrm{Ob}(D)$ on an object $c : \mathrm{Ob}(C)$ and $f(u) : D(f(x), f(y))$ on a morphism $u : C(x, y)$.

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[1] When we speak of categories, we always mean higher categories.

[2] For example, a 0-wild category has identities and composites, a 1-wild category has associativity, and a 2-wild category has pentagon coherence.

[3] In proof assistants such as Agda, we can achieve this using a variant of Licata's trick: define Cat using a record type isomorphic to $\mathcal{U}$, and similarly define $\mathsf{Cat}(C, D)$ using a record type isomorphic to the ordinary function type, but make these implementations private, and expose only the intended API. This exploits the judgmental structure of the category of types in a manner similar to [3].

Axiom 5. For $C : \mathsf{Cat}$, the function $\mathsf{Cat}(C, \mathbb{I}) \to \mathrm{Ob}(\mathbb{I})^{\mathrm{Ob}(C)}$ is (-1)-truncated. Its image is the functions $p : \mathrm{Ob}(C) \to \mathrm{Ob}(\mathbb{I})$ such that $C(x, y)$ implies $p(x) \leq p(y)$ in the standard ordering of $\{0, 1\}$.

Define for each $n : \mathbb{N}$ a category $\Delta^n : \mathsf{Cat}$, by setting $\Delta^0 = 1$ and $\Delta^{n+1} = \mathrm{Fun}(\Delta^n, \mathbb{I})$. One can construct an equivalence $\mathrm{Ob}(\Delta^n) \simeq \{0, \ldots, n\}$, such that for $C : \mathsf{Cat}$, the type $\mathsf{Cat}(C, \Delta^n)$ is equivalent to the set of functions $p : \mathrm{Ob}(C) \to \{0, \ldots, n\}$ such that $C(x, y)$ implies $p(x) \leq p(y)$.

Axiom 6 (Segal axiom)**.** The following square of categories is a pushout:

$$\begin{array}{ccc} 1 & \xrightarrow{0} & \mathbb{I} \\ {\scriptstyle 1}\big\downarrow & \ulcorner & \big\downarrow{\scriptstyle 12} \\ \mathbb{I} & \xrightarrow[01]{} & \Delta^2. \end{array}$$

Given $C : \mathsf{Cat}$ with $u : C(x, y)$ and $v : C(y, z)$, we obtain a functor $f : \Delta^2 \to C$ sending the morphisms 01, 12 to u, v, respectively. The *composite* $u \circ v : C(x, z)$ is the action of f on the "long" morphism 02.

One can prove, by induction on $n : \mathbb{N}$, that Δ^n is similarly freely generated by $n + 1$ objects and n morphisms between them. In this way one can build an associator for composition of morphisms.[4]

Say a morphism in $C(x, y)$ is invertible if it has a left inverse and a right inverse, and denote the type of invertible morphisms by $x \simeq y$. If all morphisms of C are invertible, it is a *groupoid*.

Axiom 7 (Rezk axiom)**.** For $C : \mathsf{Cat}$ with $x : \mathrm{Ob}(C)$, the type $(y : \mathrm{Ob}(C)) \times (x \simeq y)$ is contractible.

Axiom 8 (Triangulation of the square)**.** The following square of categories is a pushout:

$$\begin{array}{ccc} \mathbb{I} & \xrightarrow{02} & \Delta^2 \\ {\scriptstyle 02}\big\downarrow & \ulcorner & \big\downarrow{\scriptstyle (011,001)} \\ \Delta^2 & \xrightarrow[(001,011)]{} & \mathbb{I} \times \mathbb{I}. \end{array}$$

The following axiom is highly useful: it produces an inverse to a functor without requiring the inverse to admit an explicit description.

Axiom 9 ($\mathbb{I}$ detects equivalences)**.** Given a functor $f : \mathsf{Cat}(C, D)$, if $f \circ - : \mathsf{Cat}(\mathbb{I}, C) \to \mathsf{Cat}(\mathbb{I}, D)$ is an equivalence, then f is invertible.

Axiom 10 (Full subcategories)**.** For $C : \mathsf{Cat}$ and a family P of small propositions over $\mathrm{Ob}(C)$, we have a category C_P and functor $\iota : \mathsf{Cat}(C_P, C)$ such that for $X : \mathsf{Cat}$, the function $\iota \circ - : \mathsf{Cat}(X, C_P) \to \mathsf{Cat}(X, C)$ is an embedding, and its image is the functors $f : \mathsf{Cat}(X, C)$ such that $P(f(x))$ for $x : \mathrm{Ob}(X)$.

Axiom 11. The wild category Cat has pullbacks.

The following axiom allows us to regard types as groupoids, and to work fiberwise over them.

Axiom 12. The wild category Cat has small coproducts, and these are extensive. For $X : \mathcal{U}$, the category $DX := \coprod_X 1$ is a groupoid, and selection $X \to \mathrm{Ob}(DX)$ of coprojections is invertible.

A functor $f : E \to B$ is a *left fibration* if it is right orthogonal against $0 : 1 \to \mathbb{I}$, in the sense that the function $\mathsf{Cat}(\mathbb{I}, E) \to \mathsf{Cat}(\mathbb{I}, B) \times_{\mathsf{Cat}(1,B)} \mathsf{Cat}(1, E)$ is an equivalence.

Say that a left fibration $p : E \to B$ is *subuniversal* if for every $C : \mathsf{Cat}$, the function from $\mathsf{Cat}(C, B)$ to the type $(D : \mathsf{Cat}) \times \mathsf{Cat}(D, C)$ given by $f \mapsto (C \times_B E, \mathsf{fst})$ is (-1)-truncated, and its image is the left fibrations $D \to C$ all of whose fibers are classified by p.

Axiom 13. Every left fibration is a base change of a subuniversal left fibration.

Axiom 14. The wild category Cat has pushouts.

We hope that this formulation will allow for formalisations of higher category theory in Rocq or Agda.

[4]This requires 2-wildness of Cat, i.e., the pentagon coherator.

References

[1] Danil Annenkov, Paolo Capriotti, Nicolai Kraus, and Christian Sattler. "Two-level type theory and applications". In: *Mathematical Structures in Computer Science* 33.8 (2023), pp. 688–743.

[2] Denis-Charles Cisinski, Bastiaan Cnossen, Kim Nguyen, and Tashi Walde. *Synthetic Category Theory*. 2026.

[3] Ambrus Kaposi, András Kovács, and Nicolai Kraus. "Shallow embedding of type theory is morally correct". In: *International Conference on Mathematics of Program Construction*. Springer. 2019, pp. 329–365.