Pushforwards in Inverse Homotopical Diagrams

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Developed originally to study generalized sheaf cohomology [Bro73], Brown's theory of categories of fibrant objects has seen renewed interest in recent years coming from such disparate areas as: higher category theory [Szu17; Szu16], dependent type theory [AKL15; Shu15], and graph theory [CK24].

When applying this theory in concrete cases, one often works with categories of inverse diagrams. Namely, given a fibration category $\mathbb{C}$ and a small inverse category $\mathcal{I}$, the diagram category $\mathbb{C}^{\mathcal{I}}$ of $\mathcal{I}$-shaped diagrams in $\mathbb{C}$ is again a fibration category with fibrations defined as the *Reedy fibrations* and weak equivalences defined levelwise. Such diagram categories were studied in detail by Radulescu-Banu in [Răd09] and in the context of type theory by Shulman in [Shu15].

One can also consider $\mathcal{I}$ to carry a class of weak equivalences and ask that the diagrams $\mathcal{I} \to \mathbb{C}$ under consideration preserve this class, leading to the notion of a *homotopical* diagram. Such diagrams were used extensively by Szumiło to establish an equivalence of the homotopy theories of fibration categories and (finitely) complete quasicategories [Szu17]. In the context of dependent type theory, such diagrams have proven indispensable in several contexts, e.g., to construct path objects on the category of models of dependent type theory [KL18] and in the proof of homotopy canonicity by Kapulkin and Sattler.

A common requirement in dependent type theory is that the category of fibrant objects also be closed under pushforwards. Such categories of fibrant objects are presentations of locally cartesian closed $(\infty, 1)$-categories. Combining the two themes discussed above, we arrive at the fundamental question of this talk:

> under what conditions is the category of homotopical Reedy fibrant digrams again closed under pushforwards inside the category of all (Reedy fibrant) diagrams?

Interestingly, two extreme cases were previously established: Shulman [Shu15] showed that if *none* of the maps in $\mathcal{I}$ are weak equivalences, then pushforwards in $\mathbb{C}$ give rise to pushforwards in $\mathbb{C}^{\mathcal{I}}$; while in [KL21], the case of *all* maps being weak equivalences was also resolved in the positive. By revisiting the proof of [KL21] from the setting of models of dependent type theory, we are able to identify a fairly permissive condition on $\mathcal{I}$.

This result has applications in a variety of areas discussed above. In dependent type theory, it allows for constructions of made-to-order models of type theory, i.e., models satisfying specific conditions on its type of propositions. It is also a step towards proving that suitably defined locally cartesian closed categories of fibrant objects present the same homotopy theory as locally cartesian closed quasicategories [Kap17; Cis19].

This talk is based on joint work with Marcelo Fiore and Krzysztof Kapulkin from results in [FKL24; KL25].

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