The ∞-category of ∞-categories in simplicial type theory

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This work concerns *simplicial type theory* (STT) introduced by Riehl and Shulman [RS17]. This is an extension of homotopy type theory intended to model not only homotopy types, but ∞-category theory. Roughly, if HoTT validates the slogan "types are ∞-groupoids" then STT substantiates the slogan "some types are ∞-categories". To this end, STT extends HoTT with critical new type $\mathbb{I}$, meant to encode the category $\{0 \to 1\}$. The central axiom of STT is:

Axiom 1. *There is a type $\mathbb{I} : \mathcal{U}$ which is (1) an h-set and (2) a bounded total order with $0, 1 : \mathbb{I}$ and $\leq : \mathbb{I} \times \mathbb{I} \to \mathsf{Prop}$.*

With this type to hand, one may immediately define the notion of a synthetic morphism in a type A: it is simply a function $\mathbb{I} \to A$. One may fix the endpoints of such functions to arrive at the definition of the hom-type:

$$\hom_A(a, b) = \textstyle\sum_{f : \mathbb{I} \to A} f\,0 = a \times f\,1 = b$$

While every type in STT comes equipped with identity maps (in the form of $\mathsf{id}_a = \lambda\_ \to a$), not every type comes with a composition operation. Riehl and Shulman [RS17] show that one may capture the existence of a coherent composition operation using only a proposition in HoTT. In particular, we may use $\mathbb{I}$, the walking arrow, to specify the walking commutative triangle $\triangle$ and the walking composable pair $\wedge$. A type is called *Segal* if

$$\mathsf{isSegal}(A) = \mathsf{isEquiv}(A^{\triangle} \to A^{\wedge})$$

Furthermore, they isolate a different property, specific to the ∞-categorical case, akin to univalence: a type A is *Rezk* ($\mathsf{isRezk} : \mathcal{U} \to \mathsf{Prop}$) if every isomorphism in A comes from a path (in the HoTT sense) within A. A critical result of their paper is the following:

Theorem 1 (Riehl–Shulman)**.** *STT has models where the types which are both Segal and Rezk precisely model ∞-categories.*

STT is missing a crucial ingredient if it is to serve as a general purpose foundation of ∞-category theory: a stock of types modeling the most important and basic categories. In this work, we use an extension of STT to construct the central such type: the ∞-category of (small) ∞-categories. In addition to merely defining Cat, we verify its critical universal properties governing maps $C \to \mathsf{Cat}$ and show that they correspond to a family of categories coherently functorially parameterized by C: a cocartesian family as defined in STT by Buchholtz and Weinberger [BW23]. In particular, within this extension of STT we

1. define a particular subtype $\mathsf{Cat} \hookrightarrow \mathcal{U}$,
2. show that Cat is a category (i.e., is both Segal and Rezk),
3. show that $\mathsf{Cat} \hookrightarrow \mathcal{U}$ is the *universal cocartesian family*,
4. and, we prove a synthetic version of Lurie's celebrated straightening–unstraightening result [Lur09]: the category of functors Cat^C is equivalent to the category of cocartesian families over C.

Triangulated type theory: an extension of STT

To more precisely state our results, we must discuss the variation on STT used for these constructions: triangulated type theory [GWB24]. TTT extends base HoTT with a collection of modalities following multimodal type theory [Gra+20]. Most critical among these for our purposes is the $\flat$-modality $\langle \flat \mid - \rangle$: the global sections comonad. In our intended model, this corresponds to the groupoid core operation in category theory. To substantiate this, within the theory we postulate the following:

Axiom 2. *The counit* $\langle \flat \mid A \rangle \to A$ *is an equivalence iff* $\mathsf{const} : A \to A^{\mathbb{I}}$ *is an equivalence.*

We furthermore weaken the original axiom governing $\mathbb{I}$ from requiring a total order to merely asking for a bounded distributive lattice. This change allows us to consistently postulate the existence of an (amazing) right adjoint [Lic+18; Ril24] to $\mathbb{I} \to -$. Gratzer, Weinberger, and Buchholtz [GWB24] show that this theory is consistent with a similar model to that of STT. Moreover, within this model types that are Segal, Rezk, and *simplicial* (types A for which $\prod_{i,j:\mathbb{I}}(\mathsf{const} : A \simeq A^{i \leq j \vee j \leq i})$) correspond again to ordinary ∞-categories. Just as in op. cit., the amazing right adjoint to $\mathbb{I} \to -$ can only be applied to $\flat$-annotated types and we may only tranpose $\flat$-annotated maps, but this suffices to build a type Cat satisfying the following:

Lemma 2. *There exists a subtype* $\mathsf{Cat} \hookrightarrow \mathcal{U}$ *such that the type* $\langle \flat \mid X \to \mathsf{Cat} \rangle$ *is equivalent to* $\langle \flat \mid \sum_{A:X \to \mathcal{U}} \mathsf{isCocartFam}(A) \rangle$.

This proposition can be summarized as stating that Cat is the universal cocartesian family. Notably, we do not use any particular properties in the construction of Cat after establishing this result. We instead rely only on this universal property.

The main results

First and foremost, we show that Cat is not only the base of the universal cocartesian fibration, but itself a category.

Theorem 3. Cat *is Segal, Rezk, and simplicial.*

This theorem hinges on a result which is interesting in its own right: a characterization of morphisms in Cat:

Theorem 4 (Directed univalence)**.** *The map sending* $F : \mathbb{I} \to \mathsf{Cat}$ *to* $(F\,0, F\,1, \mathsf{cocartTransp}(F))$ *induces an equivalence:*

$$\langle \flat \mid \mathbb{I} \to \mathsf{Cat} \rangle \simeq \langle \flat \mid \textstyle\sum_{A,B:\mathcal{U}} \mathsf{isCat}(A) \times \mathsf{isCat}(B) \times A \to B \rangle$$

In other words, just as equalities in the ordinary HoTT universe coincide with equivalences, a synthetic morphism in Cat is precisely a functor. Finally, we are able to give fully synthetic proof characterizing the category $C \to \mathsf{Cat}$ following Cisinski et al. [Cis+25]:

Theorem 5 (Straightening–unstraightening)**.** *There is an embedding* $\mathsf{Cat}^C \hookrightarrow \sum_{D:\mathsf{Cat}} \hom(D, C)$ *which witnesses* Cat^C *as the subcategory of* $\mathsf{Cat}_{/C}$ *spanned by cocartesian families and cocartesian functors between them.*

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