

Univalent Multicategories

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Motivation

Multicategories were introduced by Lambek [1], [2] as a generalization of categories which allow for morphisms of multiple arity. Multicategories naturally describe mathematical objects which are parameterized over a list of “source” objects and a “destination”, such as multilinear maps or sequents. The related concept of *operad* was defined by May [3] to characterize loop spaces in algebraic topology. Some multicategories arise from a monoidal category and are called *representable*, however not all have a representing monoidal category [4].

There are several generalizations of the concept of category, such as Bénabou’s T -categories [5] and the generalized multicategories of Cruttwell and Shulman [6] which seek to describe both categories, multicategories and 2-dimensional concepts such as virtual double categories. However, none of these efforts adequately describe multicategories in univalent foundations. Both T -categories and generalized multicategories describe categories as monads in $\mathbf{Span}(\mathbf{Set})$. Multicategories are then defined by instantiating this definition for the free monoid monad on \mathbf{Set} . In univalent mathematics, this construction does not yield all categories. Instead, it yields *strict* categories whose collections of objects are mere sets and lack univalence. Furthermore, one cannot merely change \mathbf{Set} to \mathbf{Type} as types and functions in univalent mathematics do not necessarily form a category. We believe that a new approach is needed to describe multicategories in univalent foundations.

1 Distribution types

We first define a category \mathbf{Dist} of *distribution types*² which represent the shape of diagrams in a multicategory. This category consists of finite, totally ordered sets and partial monotonic functions therebetween. This approach is similar to Lurie’s construction of \mathbf{Fin}_* , which he uses to define ∞ -operads [8].

$$\begin{array}{rcccc} x & 0 & 1 & 2 & 3 \\ f(x) & 0 & 1 & 1 & \emptyset \end{array}$$

Figure 1: A function $f : 4 \rightarrow 2$ in \mathbf{Dist} .

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²Name due to Linton [7].

We interpret a partial function $f : n \rightarrow m$ as the shape of an m -ary vector of multi-morphisms whose total arity sums to at most n . For example, the function f in Figure 1 is interpreted as a tuple of one function of arity 1 and one function of arity 2, with total arity 3 (the fourth argument is unused in any morphism).

We define two important classes of morphisms. *Inert* morphisms in Dist are those which are a partial bijection. That is, an inert morphism $f : m \rightarrow n$ always consists of exactly n multi-morphisms of arity 1. We also define *active* morphisms as those which are totally defined. Inert and active morphisms form an orthogonal factorization system on Dist .

2 Multicategories

The essential data of a multicategory is given by a category displayed (in the sense of Ahrens and Lumsdaine [9]) over Dist . Given a category M displayed over Dist , we wish to interpret M_1 as a type of objects and M_n as n -ary vectors thereof. To do so, we require M to admit a cocartesian lift of every inert morphism in Dist . In particular, there always exists a lift of the morphism $\rho_i : n \rightarrow 1$ which is defined by $i \mapsto 0$. This lift of ρ_i allows us to “index” (see Figure 2) both vectors $A : M_n$ and multi-morphisms $F : \text{Hom}_{f:m \rightarrow n}(A, B)$.

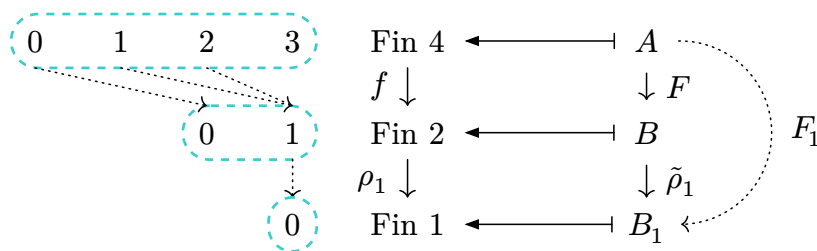


Figure 2: A multi-morphism displayed over the distribution type f (Figure 1) “indexed” by ρ_1 . The diagram on the left shows the action of f and ρ_1 on elements of the finite sets. Note F_0 has arity 1 and F_1 has arity 2.

Furthermore, this indexing operation is required to be a homotopy equivalence; i.e. we have an inverse $\langle \dots \rangle : \left(\prod_{i < n} \text{Hom}_{\rho_i \circ f}(A, B_i) \right) \rightarrow \text{Hom}_f(A, B)$ on morphisms. We also require a map from products to displayed objects $\uparrow : M_1^n \rightarrow M_n$ along with ρ_j -cocartesian morphisms from $\uparrow A$ to A_j .

2.1 Products in a multicategory

This cocartesian structure on a multicategory lets our objects $A : M_n$ behave like n -ary products of objects in M_1 . To demonstrate, consider the eta expansion of $A, \uparrow(A_1, A_2, \dots)$. We can define a morphism $\langle \tilde{\rho}_1, \tilde{\rho}_2, \dots \rangle : A \rightarrow \uparrow(A_1, A_2, \dots)$. Likewise, we have $\langle \tilde{\rho}_1, \tilde{\rho}_2, \dots \rangle : \uparrow(A_1, A_2, \dots) \rightarrow A$. We can compare morphisms based on their index, so these form an isomorphism

$$\langle \tilde{\rho}_1, \tilde{\rho}_2, \dots \rangle \circ \langle \tilde{\rho}_1, \tilde{\rho}_2, \dots \rangle = \text{id}$$

only if for all i , $(\langle \tilde{\rho}_1, \tilde{\rho}_2, \dots \rangle \circ \langle \tilde{\rho}_1, \tilde{\rho}_2, \dots \rangle)_i = \text{id}_i$, and

$$\begin{aligned}
\langle \langle \tilde{\rho}_1, \tilde{\rho}_2, \dots \rangle \circ \langle \langle \tilde{\rho}_1, \tilde{\rho}_2, \dots \rangle \rangle_i &= \langle \tilde{\rho}_1, \tilde{\rho}_2, \dots \rangle_i \circ \langle \tilde{\rho}_1, \tilde{\rho}_2, \dots \rangle \\
&= \tilde{\rho}_i \circ \langle \tilde{\rho}_1, \tilde{\rho}_2, \dots \rangle \\
&= \langle \tilde{\rho}_1, \tilde{\rho}_2, \dots \rangle_i \\
&= \tilde{\rho}_i \\
&= \text{id}_i.
\end{aligned}$$

Likewise, beta expansion $(\uparrow A)_j \simeq A_j$ holds as the domain of cocartesian morphisms is unique.

3 Results

We develop multicategories in the cubical Agda theorem prover [10] utilizing the 1Lab [11] library which already contains definitions such as bicategories, displayed categories and cocartesian lifts in cubical Agda. In addition to this (hyperlinked to the formalization), we define:

- [Dist](#) and Lurie’s category Fin_*
- [The definition of multicategory](#)
- [The bicategory of multicategories, multi-functors and multi-natural transformations](#)
- [Univalent bicategories](#) with a proof that univalent multicategories are [locally univalent](#)

3.1 Univalence

We define a univalent multicategory as a multicategory whose displayed category is univalent [9]. That is, a univalent multicategory M has univalent fiber categories M_n . As M_n is equivalent to M_1^n —an n -ary product of the underlying category M_1 of M that only consists of 1-ary morphisms—this says that the underlying category of M is also univalent.

4 Future Work

We have yet to prove several important properties of multicategories, such as whether the bicategory of multicategories is globally univalent. We also wish to show that the displayed definition of multicategory is equivalent to the classical definition with morphisms from $\text{List}(\text{Ob})$ to Ob . Finally, we have yet to define the multicategory for a monoidal category, or the adjunction between the bicategories Mon and MultiCat given by representability.

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