In Memory of Tim Lichtnau: Deligne-Mumford Stacks in Synthetic Algebraic Geometry

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Tim Lichtnau died on the 26th of December 2024. In this talk we will pay hommage by presenting the work he did for his unfinished master thesis, under our supervision. Tim was very talented and planning to start a PhD on homotopy type theory, his passing is also a great loss for the community.

He was studying higher Deligne-Mumford stacks in the context of synthetic algebraic geometry. Before presenting his work we will introduce synthetic algebraic geometry and motivate Deligne-Mumford stacks.

Synthetic algebraic geometry. The starting point of algebraic geometry is to study spaces, which are described by systems of multivariate polynomials over a general ring. These spaces are called affine schemes and they are studied using geometric techniques. For this to work we need to use a sophisticated notion of space, e.g. locally ringed spaces or Zariski sheaves.

Tim's master thesis work was part of the synthetic algebraic geometry project [CCH24; Che+24], which aims to develop algebraic geometry using HoTT extended by axioms. In this alternative approach we assume a base ring R (which is assumed to be a set). The space corresponding to e.g. a single polynomial $P : R[X]$ is simply defined as the type $\{x : R \mid P(x) = 0\}$, without any extra structure. This works because a type will ultimately be interpreted as a Zariski sheaf.

We assume the duality axiom, which implies e.g. that any map in $R \to R$ is a polynomial. We also assume that R is local and that affine schemes satisfy a weakening of the axiom of choice called Zariski local choice.

These axioms are satisfied by the interpretation of HoTT in the higher Zariski topos, where sets are interpreted as Zariski sheaves. So we can interpret any result about sets in synthetic algebraic as a result about Zariski sheaves. The interpretation also works for higher types, which are interpreted as somewhat more exotic objects. We give a quick summary of the terminology:

HoTT	Traditional name	Intuition
Sets	Zariski sheaves	Sheaves of sets on the Zariski site
Groupoids	Zariski stacks	Sheaves of groupoids on the Zariski site
Types	Higher Zariski stacks	Sheaves of ∞-groupoids on the Zariski site

In this abstract we will also use higher étale stack, i.e. sheaves of ∞-groupoids on the étale site. They can be seen as higher Zariski sheaves (i.e. types) satisfying a certain proposition.

Deligne-Mumford stacks. While algebraic geometry starts with the study of affine schemes, it quickly demands more general spaces. Schemes are defined synthetically as sets that are Zariski-locally affine. The most famous examples are the projective spaces $\mathbb{P}^n$, which are defined synthetically by $\mathbb{P}^n = \{L \subset R^{n+1} \mid L \text{ is a line}\}$.

Moduli spaces are spaces of geometric objects and play an important role in algebraic geometry — but they rarely are schemes. For example let us consider the type of smooth cubic curves. Since some smooth cubic curves have non-trivial automorphisms, this type is not a set but rather a groupoid, so it cannot be a scheme.

This motivated the notion of Deligne-Mumford stacks. Those are étale stacks which are in some sense not too far from being schemes. Examples include the aforementioned moduli stack of smooth cubic curves, or more interestingly the moduli stack of curves of a given genus g with n marked points [Sil09; Ols23]. In synthetic algebraic geometry, being a Deligne-Mumford stack will simply mean being a 1-type satisfying some proposition.

This was later extended to higher Deligne-Mumford stacks, i.e. sheaves of ∞-groupoids on the étale site which are not too far from being schemes [Sim96].

Synthetic higher Deligne-Mumford stacks. Now we will present Tim's strikingly elegant definition of higher Deligne-Mumford stacks. This presentation relies on two crucial characteristics of HoTT:

1. Lex modalities are well behaved in HoTT, so that defining higher étale stacks is no more difficult than defining étale sheaves.
2. Types have a natural higher homotopical structure, so defining higher Deligne-Mumford stacks is no more difficult than defining Deligne-Mumford sets (aka algebraic spaces).

We will drop the "higher" as we consider arbitrary types by default. We are now ready for the definition. The proposition $\epsilon = 0$ below can be thought of as a closed dense subspace of the point. Recall that an affine scheme merely is the set of zeros of polynomials $P_1, \ldots, P_l : R[X_1, \ldots, X_n]$.

Definition 1. *A type X is formally étale if for all $\epsilon : R$ such that $\epsilon^2 = 0$, we have that $X \to X^{\epsilon=0}$ is an equivalence.*

Definition 2. *We define $\mathbb{T}$ as the type of formally étale non-empty affine schemes.*

Definition 3. *A type X is an étale stack if and only if for all $S : \mathbb{T}$ we have that the map $X \to X^{\|S\|}$ is an equivalence.*

Definition 4. *The class of covering stacks is the smallest class of étale stacks containing $\mathbb{T}$ such that if there exists $S : \mathbb{T}$ and a map $S \to X$ whose fibers are covering stacks, then X is a covering stack.*

Definition 5. *An étale stack X is a Deligne-Mumford stack if there exists an affine scheme S and a map $S \to X$ whose fibers are covering stacks.*

If a Deligne-Mumford stack is covering, then it is non-empty and formally étale. We expect the converse to hold. With this definition Tim proved the following:

1. Deligne-Mumford stacks are stable under Σ and identity types.
2. Deligne-Mumford stacks are stable under covering quotients (i.e. given an étale stack Y and a Deligne-Mumford stack X with $p : X \to Y$ whose fibers are covering stacks, we have that Y is a Deligne-Mumford stack).
3. Deligne-Mumford stacks have étale descent (i.e. the type of Deligne-Mumford stack is itself an étale stack).

Algebraic spaces. Tim then turned his attention to studying examples of algebraic spaces (i.e. Deligne-Mumford stacks that are sets). We will present his chain of strict inclusions:

$$\{\text{Schemes}\} \subsetneq \{\text{Locally separated algebraic spaces}\} \subsetneq \{\text{Algebraic spaces}\}$$

References

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