Elementary ∞-toposes from type theory

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Since its conception, it has been speculated (see e.g. [9]) that Homotopy Type Theory (HoTT) [12] is the internal language of particular higher categories also called *elementary ∞-toposes*. A precise formulation of this statement was given by Kapulkin and Lumsdaine in [4, Conj. 3.7] and is known as the *internal language conjecture*. Here, HoTT is understood as Martin-Löf dependent type theory with dependent sums, dependent products, intensional identity types and univalent universes.

The conjecture fits into a series of important steps that have been made so far towards establishing such a connection between higher categories and type theory. Kapulkin and Szumiło [5] showed that dependent type theory with intensional identity types is the internal language of finitely complete ∞-categories. Assuming in addition dependent products, Kapulkin [3] proved that every model of such a type theory presents a locally cartesian closed ∞-category. Shulman [11] showed that HoTT can be interpreted as internal language into any Grothendieck ∞-topos.

Proving a correspondence between HoTT and elementary ∞-toposes is currently an open problem. Elementary ∞-toposes are supposed to generalise both Grothendieck ∞-toposes as studied by Lurie [6] and ordinary 1-toposes, whose internal language is a version of intuitionistic higher-order logic [7]. There is currently no generally agreed upon definition of an elementary ∞-topos, even though there has been increasing interest in the topic in the last decade. Two important proposals for a definition were made by Shulman in [10] and by Rasekh in [8].

We define an elementary ∞-topos as a finitely complete, locally cartesian closed ∞-category with enough univalent morphisms. In its precise statement, the internal language conjecture then asserts that there is a Dwyer-Kan equivalence

$$\mathrm{Cl}_\infty \colon \mathbf{CxlCat}_{\mathtt{HoTT}} \xrightarrow{\sim} \mathbf{ElTop}_\infty$$

between the category of categorical models of HoTT and the category of elementary ∞-toposes induced by sending each model to its ∞-localisation at the class of homotopy equivalences. Thus, Cl_∞ takes a model of HoTT and sends it to an ∞-category that is obtained by inverting all homotopy equivalences in that model. Proving that this functor exists, i.e. that the ∞-localisation takes values in $\mathbf{ElTop}_\infty$, and showing that this is a Dwyer-Kan equivalence has so far been an open problem.

In the talk, we will present the work [1] in which we prove the existence of this functor Cl_∞, a first step towards proving the conjecture, as well as ongoing work in which this result is generalised and where we consider also higher inductive types and colimit types. Concretely, we show that every model of HoTT presents an elementary ∞-topos via its ∞-localisation. This is achieved in two steps. First, we use the fact that every model of HoTT has the structure of a tribe in the sense of Joyal [2]. We extend Joyal's theory of tribes by introducing the notion of a univalent fibration in a tribe and the notion of a univalent tribe. In particular, every categorical model of HoTT is such a univalent tribe. In the second step, we prove that every univalent tribe presents via its ∞-localisation an elementary ∞-topos. Thus, the functor Cl_∞ can be obtained as a composite:

$$\mathbf{CxlCat}_{\mathtt{HoTT}} \to \mathbf{UnivTrb} \to \mathbf{ElTop}_{\infty}.$$

Unlike our definition, the definitions of elementary ∞-toposes in [10] and [8] also assume the existence of a subobject classifier and finite colimits.

In [1], we show that the axioms of finite completeness, local cartesian closedness and the existence of univalent morphisms imply the existence of a subobject classifier for each univalent morphism, classifying the monomorphisms that are classified by that univalent morphism. The reason why we do not demand there to be a single subobject classifier for *all* monomorphisms is that in type theory an object of propositions is usually as large as the ambient universe (See [12, §10.1.4]). Thus, one needs to make an additional assumption also known as the *axiom of propositional resizing* ([12, Axiom 3.5.5]) in order to downsize that object of propositions. We show that if the type theory does satisfy the axiom of propositional resizing, then the ∞-localisation of any of its categorical models has indeed a single subobject classifier for all monomorphisms.

Moreover, in ongoing work, we prove that the existence of pushout types as higher inductive types and empty types in the type theory implies the existence of finite colimits in the ∞-localisation.

References

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