

# The structure homomorphism principle in simplicial type theory

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**Introduction** A notable consequence of univalence is that it allows for a widely applicable characterisation of identities as isomorphisms through the structure identity principle [Ahr+25; CD13; Uni13]. Directed type theories [ANW23; Kav19; LH11; Neu25; NA24; Nor18; Nuy15; Nuy20; RS17; War13; WL20; Wea24] seek to add a directed analogue to the identity type to type theory. There is a meaningful directed analogue to univalence. One is then naturally lead to wonder if there is a directed analogue [WL20] to the structure identity principle that could similarly characterise the directed analogue of the identity type as structure-preserving homomorphisms? We explore the answer to this question — the structure homomorphism principle within simplicial type theory.

**Synthetic categories in simplicial type theory** We shall work in simplicial type theory [RS17]. Among other things, simplicial type theory extends HoTT with the set  $\mathbb{I}$  equipped with a bounded total order:

**Axiom 1.** *We have a set  $\mathbb{I}$  equipped with two points  $0 : \mathbb{I}$  and  $1 : \mathbb{I}$  such that  $0 \leq 1$  holds.*

**Definition 2.** For  $a : X$  and  $b : X$ , the type of morphisms  $\text{hom}_X(a, b)$  is given by:

$$\text{hom}_X(a, b) \triangleq \sum_{p : \mathbb{I} \rightarrow X} p(0) = a \times p(1) = b$$

Differing from Riehl and Shulman [RS17], we keep  $\mathbb{I}$  as an ordinary type. The aforementioned type of morphisms  $\text{hom}_X(a, b)$  is precisely the directed analogue to the identity type. For an arbitrary type  $X$ , one cannot necessarily compose compatible morphisms. In particular, the  $\text{hom}_X(-, -)$  type need not be well-behaved, that is, it need not make  $X$  an  $(\infty)$ -category. Instead, we must isolate types that do allow for composites by stating the **Segal** condition. We must also ask that  $X$  be **Rezk** — that synthetic isomorphisms  $\text{Iso}(x, y)$  be equivalent to the identity type  $x =_X y$ .

**Definition 3.** A **category** is a type that is both Rezk and Segal. A **groupoid** is a category where every morphism is an isomorphism.

The quintessential example of a category is  $\mathcal{S}$  — the category of spaces.  $\mathcal{S}$  is the subtype of the universe such that a  $A :_{\mathfrak{b}} X \rightarrow \mathcal{U}$  factors through  $\mathcal{S}$  iff  $A$  is covariant [RS17, Section 8]. Note that  $A$  being a  $\mathfrak{b}$ -family is essential here, where  $\mathfrak{b}$  is the groupoid core modality from Gratzer et al. [GWB24]. Roughly speaking,  $\mathfrak{b}$  gets rids of all the “higher cells” and only retains the points.

**Theorem 4.**  *$\mathcal{S}$  is directed univalent. We have that the canonical map  $\text{mortofun} : (\mathbb{I} \rightarrow \mathcal{S}) \rightarrow \sum_{A, B : \mathcal{S}} A \rightarrow B$  is an equivalence. Consequently, the type  $\text{hom}_{\mathcal{S}}(A, B)$  is equivalent to  $A \rightarrow B$ .*

Ordinary function composition and the ordinary univalence axiom makes  $\mathcal{S}$  Segal and Rezk, respectively. We can also isolate full subcategories  $\mathcal{S}_{\leq n} \hookrightarrow \mathcal{S}$  whose objects are  $n$ -truncated.

**The structure homomorphism principle** In ordinary HoTT,  $\sum$ -types commute with the identity type. This allows us to derive a useful structure identity principle for algebraic structures. In our directed setting, we have a similar commutation: a function  $\mathbb{I} \rightarrow \sum_{a:A} B(a)$  decomposes into a pair  $\sum_{a:\mathbb{I} \rightarrow A} \prod_{i:\mathbb{I}} B(a(i))$ . This leads to the following:

**Definition 5.** The structure homomorphism principle (SHP) is an informal slogan that synthetic morphisms correspond to structure-preserving functions, that is, homomorphisms.

We shall present a few examples of the SHP in action. Consider the type of magmas:  $\sum_{X:\text{Set}} X \times X \rightarrow X$ . This type is not a category, and thus we cannot say much about its synthetic morphisms. If we restrict ourselves to magmas where the carrier is in the directed univalent  $\mathcal{S}_{\leq 0}$ , then we do get a category  $\sum_{X:\mathcal{S}_{\leq 0}} X \times X \rightarrow X$  and we can show:

**Example 6.** For any magmas  $(M, \cdot_M)$  and  $(N, \cdot_N)$ , we have an equivalence:

$$\text{magmaSHP} : \text{hom}((M, \cdot_M), (N, \cdot_N)) \simeq \sum_{f:M \rightarrow N} (\cdot_N) \circ (f \times f) = f \circ (\cdot_M)$$

Notice that the RHS of the equivalence above is precisely a magma homomorphism. Indeed, one can derive a more general theorem for algebraic structures that encompasses monoids, groups, rings, et cetera:

**Theorem 7.** *The type of models of an algebraic theory with carriers in  $\mathcal{S}_{\leq 0}$  is a category whose synthetic morphisms are equivalent to ordinary homomorphisms.*

SHP is not limited to algebraic theories. We can also show an SHP for the type of precategories, which is the type of models of an *essentially* algebraic theory:

$$\text{Precat} \triangleq \sum_{\text{Ob}:\mathcal{S}_{\leq 0}} \sum_{\text{Mor}:\text{Ob} \times \text{Ob} \rightarrow \mathcal{S}_{\leq 0}} \sum_{\text{id}:\prod_{x:\text{Ob}} \text{Mor}(x,x)} \dots$$

**Theorem 8.** *Precat is a category whose objects are precategories and whose morphisms correspond to functors.*

**Agda formalisation** We are working on a formalisation of our STT proofs in Cubical Agda [VMA21] with much of the simplicial type theory (such as the interval) postulated. Our formalisation is intended to match the theory presented in Gratzner et al. [GWB24]. Thus, we do not rely on definitional rewrite rules, and minimise the usage of modalities in our work, using only Agda’s `---cohesion` flag [Vez18]. We take advantage of the work of de Jong et al. [dKL26]. Their formalisation contains key lemmas about orthogonality that are particularly useful for our work.

We have also chosen not to use the Rzk proof assistant [Kud23; KRW24]. Rzk is based on STT, but our work requires the interval to be a first-class type, whereas Rzk’s  $\mathbb{I}$  is a pre-type. Rzk can also not work with the modalities we need to extend HoTT with to correctly reason about  $\mathcal{S}$ .

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