# Symmetric products in HoTT

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**Classical definition** For a CW-complex $X$ we define the $n$-th symmetric product of $X$ as $SP^n X = X^n/\Sigma_n$, where $\Sigma_n$ is the $n$-th symmetric group, acting on $X^n$ by permutation of coordinates. This definition doesn't translate well to HoTT, because adding in paths $(x_1, \dots) = (x_{\sigma(1)}, \dots)$ leads to loops, getting rid of which creates an infinite tower of coherences. A more homotopy-invariant definition of $SP^n$ is as a colimit of a diagram with one space $X^n$ with morphisms to itself corresponding to every permutation. In terms of the universal property, $SP^n X$ is the space through which every fully commutative function from $X^n$ factors uniquely. This definition, albeit not yet a full construction, is more workable in HoTT, provided that we choose an appropriate notion of a fully commutative function.

**Commutativity in HoTT** Commutativity of functions is expressed in HoTT via the notion of a commutativity structure, introduced in [Bru16] and later expanded upon in [LW25]. We say that a commutativity structure on a function $f : A^2 \to B$ (where $A^2 :\equiv \text{Bool} \to A$) is a function $c : \prod_{X:B\Sigma_2}(X \to A) \to B$ with $c(\text{Bool}) = f$, where $B\Sigma_2$ is the type of all 2-element types. However, the type of all commutativity structures doesn't correspond to the classical understanding of the space of all commutative functions from $A^2$ to $B$, because it contains extra information. Consider $A = *$, then every function $f : A^2 \to B$ is commutative, so we would expect the type of commutative functions from $A^2$ to $B$ to be $A^2 \to B$, which is just $B$. However the type of commutativity structures here is $B\Sigma_2 \to B$. The underlying problem is that the extended diagonal map $\Delta_c(X, x) = c(X, \lambda\_.x)$ may be non-trivial, while we would expect the structure obtained by swapping $x$s in $f(x, x)$ to trivialize. This motivates considering only commutative functions that are constant on the extended diagonal when checking universal properties.

**Borel construction** As stated in [BDR18], the type of commutativity structures of $A^n \to B$ is the type of homotopy fixpoints of the operation of flipping arguments. Therefore, by currying, the type $\sum_{X:B\Sigma_n} X \to A$ is the homotopy quotient $A^n//\Sigma_n$, also called the Borel construction $E\Sigma_n \times_{\Sigma_n} A^n$. This space behaves like the regular quotient whenever the group action is free. Since the action of $\Sigma_n$ is not free on fat diagonals (subspaces of $A^n$ where at least one coordinate is equal to the other), we need to correct these diagonals of the Borel construction to get the symmetric product. We adapt our method for that from [Far96].

**Second symmetric product** We will try to provide a classical definition of the second symmetric product of a CW-complex $X$ that translates well to HoTT. Let $d : B\Sigma_2 \times X \to E\Sigma_2 \times_{\Sigma_2} X^2$ be the diagonal map $([b], x) \mapsto [b, (x, x)]$ (this is well defined, because for $[b] = [b']$ we get $[b, (x, x)] = [b', (x, x)]$). Then let $S^2(X)$ be the homotopy pushout of the following diagram:

$$\begin{array}{ccc} B\Sigma_2 \times X & \xrightarrow{\ d\ } & E\Sigma_2 \times_{\Sigma_2} X^2 \\ \Big\downarrow \text{proj}_2 & & \\ X & & \end{array}$$

Let's analyze the fibers of the natural map $\Phi : S^2(X) \to SP^2 X$. Over $[x, x]$ the fiber is a cone over $B\Sigma_2$ and therefore contractible, and over $[x, y]$ where $x \neq y$ the fiber is $E\Sigma_2$ by the freeness of the action of $E\Sigma_2$ on $(x, y)$. Since appropriate diagonal maps are cofibrations, by 4K.2 in [Hat02] $\Phi$ is a homotopy equivalence. Since homotopy pushouts are definable in HoTT, this gives us the following definition of the second symmetric product:

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SP2 (A : Type)
  | sigma : (f : Σ (X : BSigma_2) (X → A)) → SP2 A
  | delta : A → SP2 A
  | tau : Π (a : A) (X : BSigma_2) → delta a = sigma (X, λ _. a)
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We can check that this definition satisfies the universal property with regards to commutativity structures constant on the extended diagonal. This construction has also been described by Ulrik Buchholtz in an unpublished work. He proved that it maps sets to sets, and conjectured that it would in general map $n$-types to $n$-types. That conjecture however is inconsistent with classical topology, because $SP^2(B\Sigma_2)$ has the fundamental group $\Sigma_2$, but doesn't have the cohomology of $B\Sigma_2$, and therefore is not a groupoid.

**Higher symmetric products** Consider, in classical algebraic topology, the map $\phi : E\Sigma_n \times_{\Sigma_n} X^n \to SP^n X$. Its fiber over a point $[x_1, \ldots, x_1, x_2, \ldots, x_2, \ldots]$ is determined by how many times each coordinate occurs in the point (if $x_i$ appears $k_i$ times, then the fiber is $\cdots \times B\Sigma_{k_i} \times \cdots \subseteq B\Sigma_n$). These classes of points correspond to conjugacy classes in $\Sigma_n$. Our strategy for constructing $SP^n X$ from $E\Sigma_n \times_{\Sigma_n} X^n$ in HoTT will still be to contract such fibers, but we need to take special care for intersecting cases. For example, let's consider a space constructed from $E\Sigma_3 \times_{\Sigma_3} X^3$ by adding mapping cones $B\Sigma_2 \ni b \hookrightarrow [b \sqcup *, (x, x, y)]$ and $B\Sigma_3 \ni t \hookrightarrow [t, (x, x, x)]$ for every $x, y \in X$, analogously to the construction of $S^2(X)$. The subspace corresponding to $[x, x, x]$ in this space is a cone over $B\Sigma_3$ glued along $B\Sigma_2 \subseteq B\Sigma_3$ to a cone over $B\Sigma_2$, which makes it equivalent to $\Sigma(B\Sigma_2)$ and therefore not contractible.

The solution is adding in a cone not only over the points from $E\Sigma_n \times_{\Sigma_n} X^n$, but over the whole structure of contractions obtained from dealing with more general cases. We reason by induction on the poset of conjugacy classes in $\Sigma_n$, and construct our contractions step-by-step. This can be realized by an iterated pushout construction, for example we define $SP^3 X$ as $S^3(X)$ resulting from the following homotopy pushout diagrams:

$$
\begin{array}{ccc}
B\Sigma_2 \times X^2 & \xrightarrow{\psi} & E\Sigma_3 \times_{\Sigma_3} X^3 \\
\downarrow{\scriptstyle \mathrm{proj}_{X^2}} & & \downarrow \\
X^2 & \longrightarrow & S^3_2(X)
\end{array}
\qquad
\begin{array}{ccc}
X \times (B\Sigma_3 // B\Sigma_2) & \xrightarrow{\varphi} & S^3_2(X) \\
\downarrow{\scriptstyle \mathrm{proj}_{X}} & & \downarrow \\
X & \longrightarrow & S^3(X)
\end{array},
$$

where $\psi$ is the map lifted from $B\Sigma_2 \to B\Sigma_3$, $B\Sigma_3 // B\Sigma_2$ is a mapping cone of the inclusion, $\varphi(x, \mathrm{in}[t]) = [t, (x, x, x)]$, and $ap_{\varphi(x,\cdot)}(\mathrm{glue}\, b) = \mathrm{glue}(b, (x, x))$. For higher symmetric products, we build the approporiate mapping cones of classifying spaces along with intermediate constructions of the product, and write analogous pushouts.

**The Dold-Thom theorem** The most important application of the symmetric products in classical homotopy theory is the Dold-Thom theorem (see [Hat02], 4.K), which states that the homotopy groups of $SP^\infty X = \mathrm{colim}_n\, SP^n X$ are isomorphic to the corresponding reduced (cellular) homology groups of $X$. The main obstacle of the proof of this theorem is in proving that the function $SP^\infty X \xrightarrow{SP^\infty \pi} SP^\infty(X/A)$ is a quasifibration. Homology axioms then follow from the homotopy fiber sequence, because the fiber of $SP^\infty \pi$ is $SP^\infty A$. This obstacle disappears in HoTT, since the homotopy fiber sequence works for every definable function. The proof that homotopy groups of $SP^\infty X$ define a homology theory is therefore reduced to showing that the fiber of $SP^\infty \pi$ is $SP^\infty A$. This may lead to a more computationally viable definition of homology than the currently used definition of [Gra18].

**Formalization** The relevant proofs have been formalized in Arend at https://github.com/WPaupa/symmetric-products.

## References


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