

A General Cubical Framework for Coherence Theorems

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This work is motivated by the study of stronger variants of type theories obtained by adding new definitional equalities, or weaker variants obtained by replacing definitional equalities by propositional equalities. The goal is to understand how to compare a weak type theory with a stronger type theory.

From a logical perspective, we want to establish *conservativity*: the fact that any statement proven in the stronger type theory was already provable in the weaker theory. From an algebraic and homotopy-theoretic point of view, we want to prove a *coherence* or *strictification* theorem: any model of the weak theory can be shown to be equivalent to some model of the strong theory, for the appropriate notion of equivalence. More formally, we want to prove that two type theories are *Morita equivalent*, a notion introduced by Isaev (2018),

The situation is quite well understood for type theories satisfying Uniqueness of Identity Proofs (UIP). The conservativity of Extensional Type Theory (ETT) over Intensional Type Theory (ITT; with UIP and function extensionality) was proven by Hofmann (1995), and further studied by Oury (2005), Winterhalter, Sozeau, and Tabareau (2019), Kapulkin and Li (2025). Terms of ETT are translated to terms of ITT by replacing implicit conversions over definitional equalities by explicit transports over propositional equalities. UIP ensures that the choices of propositional equalities won't matter in the transports.

We want to be able to prove similar results for type theories without UIP, such as variants of HoTT. That conservativity results could be proven without UIP was first conjectured by Altenkirch et al. (2017), focusing on the computation rule of the J -eliminator for identity types.

In the absence of UIP, we follow the same strategy of replacing implicit conversions by explicit transports over homotopies. However, we cannot transport over arbitrarily chosen homotopies, we need to identify a notion of *coherent* homotopy, and only transport over *coherent* homotopies. In previous work (Bocquet 2023b), I have proposed a definition of these *coherent* homotopies, but lacked the technical means to make further progress. In this talk, I will revisit the same ideas using cubical methods.

General setup Generalizing from type theory, we can consider arbitrary Generalized Algebraic Theories. We assume that we have a GAT \mathcal{T}_w (the *weak* theory) and an equational extension \mathcal{T}_s (the *strong* theory). We also assume that \mathcal{T}_w has a homotopy theory, and that this transfers to \mathcal{T}_s .

The equational extension should be specified by a collection H of right homotopies in \mathcal{T}_w , that are to be made constant in \mathcal{T}_s . For example, in the case of the extension of weak monoidal categories to strict monoidal categories, we would consider the three homotopies

$$\begin{aligned} (x, y, z : \mathbf{Ob} \vdash \alpha_{x,y,z} : (x \otimes (y \otimes z)) &\cong ((x \otimes y) \otimes z)), \\ (x : \mathbf{Ob} \vdash \lambda_x : (1 \otimes x) &\cong x), \\ (x : \mathbf{Ob} \vdash \rho_x : (x \otimes 1) &\cong x), \end{aligned}$$

along with additional higher homotopies for the pentagon and triangle identities¹.

¹ H actually needs to be a category of homotopies, so as to specify how the pentagon and triangle identities depend on the associator and unitors.

The goal is to prove that the unique map $\eta : \mathbf{0}_w \rightarrow \mathbf{0}_s$ between the initial models of the weak and strong theories is a weak equivalence². The components of $\mathbf{0}_w$ have structures of ∞ -groupoids, with homotopies and higher homotopies as the 1- and higher cells. The *coherent* homotopies should correspond to smaller ∞ -groupoid structures on these components. Informally, they should be generated from the homotopies in H , the ∞ -groupoid operations, and congruence with respect to the operations of \mathcal{T}_w . Coherence should correspond to these ∞ -groupoids being 0-truncated.

In order to make this formal, we need to have an explicit notion of “higher \mathcal{T}_w -algebra”, i.e. a \mathcal{T}_w -algebra whose components are ∞ -groupoids. Building on ideas from previous work (Bocquet 2023a), cubical \mathcal{T}_w -algebras, i.e. \mathcal{T}_w -algebras internal to cubical sets, provide a suitable approximation of the notion of higher \mathcal{T}_w -algebra. An adjunction $(\Delta \dashv 1_{\square}^*)$ allows one to compare ordinary and cubical \mathcal{T}_w -algebras. The *strict Rezk completion* is then an important tool when working with cubical algebras; it is a cubical \mathcal{T}_w -algebra satisfying a completeness condition, providing an equivalence between cubical paths and homotopies.

A factorization in cubical algebras The map $\eta : \mathbf{0}_w \rightarrow \mathbf{0}_s$ can be seen as a map $\Delta\eta : \Delta\mathbf{0}_w \rightarrow \Delta\mathbf{0}_s$ between discrete cubical algebras. We consider a factorization among cubical \mathcal{T}_w -algebras

$$\Delta\mathbf{0}_w \rightarrow \mathbf{0}_c \rightarrow \Delta\mathbf{0}_s.$$

The cubical algebra $\mathbf{0}_c$ is defined as the initial object among cubical \mathcal{T}_w -algebras equipped with:

- **Fibrant components**; meaning that the components of $\mathbf{0}_c$ can be seen as ∞ -groupoids.
- **Cubical lifts** of the homotopies in H , meaning that for every homotopy $(\Gamma \vdash h : x \simeq y) \in H$, we add a cubical path $(\Gamma \vdash \tilde{h} : (y, h) \sim (x, r(x)))$, where r is the constant homotopy.

The situation can be summarized as follows:

- In $\mathbf{0}_w$, the homotopies in H are just homotopies (**weak**).
- In $\mathbf{0}_c$, the homotopies in H become **cubical** paths.
- In $\mathbf{0}_s$, the homotopies in H become **strict** equalities.

The general coherence theorem

Claim. *If the following conditions are satisfied:*

1. *The cubical \mathcal{T}_w -algebra $\Delta\mathbf{0}_w$ has a **strict Rezk completion**.*
2. *The cubical \mathcal{T}_s -algebra $\Delta\mathbf{0}_s$ has a **strict Rezk completion**.*
3. *The components of $\mathbf{0}_c$ are **0-truncated**.*

Then $\eta : \mathbf{0}_w \rightarrow \mathbf{0}_s$ is a weak equivalence.

The strict Rezk completion of $\Delta\mathbf{0}_w$ is used to go from $1_{\square}^*(\mathbf{0}_c)$ to $\mathbf{0}_w$. The components of $\mathbf{0}_c$ being 0-truncated is used to go from $\mathbf{0}_s$ to $1_{\square}^*(\mathbf{0}_c)$. Combining the above two points, we can go from $\mathbf{0}_s$ to $\mathbf{0}_w$. The strict Rezk completion of $\Delta\mathbf{0}_s$ is used to understand what happens after going back to $\mathbf{0}_s$.

Strict Rezk completions can be constructed following Bocquet (2023a), or using the more general construction of Sattler (2023). In planned applications to type theory, the components of $\mathbf{0}_c$ being 0-truncated should be a consequence of normalization (up to cubical paths) for $\mathbf{0}_c$. This is similar to the normalization and coherence results of Uemura (2022) for ∞ -type theories.

²More generally, we look at the components of the unit of the adjunction between $\mathbf{Alg}(\mathcal{T}_w)$ and $\mathbf{Alg}(\mathcal{T}_s)$.

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