Projective Presentations of Lex Modalities

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Motivation

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Goal

Work with subtopoi in HoTT in a sheaf theoretic way.

Subtopoi in HoTT

Consider a family of propositions $P: I \to \operatorname{Prop}_{\mathcal{U}}$

Definition (1)

A type X is a **sheaf** for P if for all i : I the natural map

$$X \to (P(i) \to X)$$

is an equivalence. We define $U_P := \{X : U \mid X \text{ is a sheaf}\}.$

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Definition $(^3)$

The choice of a subuniverse and sheafification functor, such that there exists a family of propositions generating it is called a **topological modality**.

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• Trivial presentation $T = \{1\}$, presenting whole universe.

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- ► Given any topological modality, defined by P : I → Prop_U, the Σ-closure of P ∪ 1 gives a presentation.

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More interesting examples will need new axioms in HoTT...

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⁴Gratzer, Weinberger, and Buchholtz 2024.

Example

 $\mathcal{T}_{\mathrm{fppf}} := \{\{x \in R \mid g(x) = 0\} \mid g \text{ a monic polynomial in } R\}$

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Simplicial / triangulated type theory (TTT)⁴: Adds a bounded distributive lattice $\mathbb I$ to HoTT + axioms.

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 $T_{\text{simp}} := \Sigma - \text{closure}(\{(i \le j) + (j \le i) \mid i, j \in \mathbb{I}\})$

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Find easier way to determine which types are sheaves.

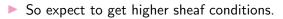
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Set / 0-type	Sheaf of sets
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- So expect to get higher sheaf conditions.
- For each *n*, want a condition for an *n*-type to be a sheaf.

Definition

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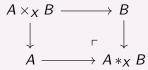
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Lemma

- Any equivalence is a cover.
- Covers are closed under pullback and composition.

Definition

Given $f : A \rightarrow X$ and $g : B \rightarrow X$ their **join** is the pushout



Given $f : A \to X$ write A_X^{*n} for the *n*-fold iterated join of *f* with itself.

Fix a presentation T.

Theorem (Sheaf Condition)

Let X be an n-type. Then X is a sheaf for T iff for all T-covers $f : A \rightarrow B$ the natural map

$$(B \rightarrow X) \rightarrow (A_B^{*n+2} \rightarrow X)$$

is an equivalence.

Question

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Corollary

A 0-type X is a sheaf for T iff for all T-covers $f : A \rightarrow B$ the natural map

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Proof.

Using the sheaf conditions + the axioms of TTT ${\mathbb I}$ is a sheaf iff

$$\mathbb{I} \simeq \lim (\mathbb{I}/(i \le j) \times \mathbb{I}/(j \le i) \rightrightarrows \mathbb{I}/(i = j))$$

Pure algebra!

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Example

Both of the running examples in SAG / TTT are projective (at least in models).

Lemma (*T*-local choice)

Let T be a projective presentation and C be projective. Let $B: C \to U$ be such that $\prod_{c:C} \bigcirc_{\mathcal{T}} ||B(c)||$. Then there is a T-cover $f: Z \to C$ such that

 $\prod_{z:Z} B(f(z))$

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Goal

How does cohomology descend to modalities from projective presentations?

Definition

Given a type $X:\mathcal{U}_{\mathcal{T}}$ and a local group $G:\mathcal{U}_{\mathcal{T}},$ its cohomology is given by

$$H^n_T(X,G) := \bigcirc_T ||X \to \bigcirc_T K(G,n)||_0$$

$$A^X \xrightarrow{d^0} A^{X \times X} \xrightarrow{d^1} A^{X \times X \times X}$$

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$$d^{0}(f)(x,x') := f(x) - f(x')$$

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$$d^{1}(f)(x, x', x'') := f(x, x') - f(x, x'') + f(x', x'')$$

$$A^X \xrightarrow{d^0} A^{X \times X} \xrightarrow{d^1} A^{X \times X \times X}$$

Given a presentation T we will say A satisfies **descent** for T if for all $X \in T$ the above sequence is exact.

Let T be a projective presentation and A an abelian group sheaf satisfying descent for T. Then for all projective X, we have $H^1_T(X, A) = 0$, where 0 is the trivial abelian group.

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► Take $\chi : X \to \bigcirc_T K(A, 1)$. χ is *T*-locally merely constant.

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Example (for algebraic geometry enthusiasts)

In SAG, an important class of modules (*quasi-coherent*) satisfy descent for $T_{\rm fppf}$. Hence these have 0 cohomology on projectives - including *R* (and all *affine schemes*).

Thank you!

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