Projective Presentations of Lex Modalities

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Goal

Work with subtopoi in HoTT in a sheaf theoretic way.

Subtopoi in HoTT

Consider a family of propositions $P : I \to \mathrm{Prop}_{\mathcal{U}}$

Definition ([1])

A type X is a **sheaf** for P if for all $i : I$ the natural map

$$X \to (P(i) \to X)$$

is an equivalence. We define $\mathcal{U}_P := \{X : \mathcal{U} \mid X \text{ is a sheaf}\}$.

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Definition ([3])

The choice of a subuniverse and sheafification functor, such that there exists a family of propositions generating it is called a **topological modality**.

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Presentations

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More interesting examples will need new axioms in HoTT...

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Interesting Examples

(Presheaf) synthetic algebraic geometry (SAG)[3]: Adds a ring R to HoTT + axioms.

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$T_{\mathrm{fppf}} := \{\{x \in R \mid g(x) = 0\} \mid g$ a monic polynomial in $R\}$

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$T_{\mathrm{simp}} := \Sigma\text{−closure}(\{(i \leq j) + (j \leq i) \mid i, j \in \mathbb{I}\})$

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- So expect to get higher sheaf conditions.
- For each n, want a condition for an n-type to be a sheaf.

Basic properties

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A **cover** for a presentation T is a map $f : X \to Y$ such that for all $y : Y$, the fiber $f^{-1}(y)$ is in T.

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Lemma

- *Any equivalence is a cover.*
- *Covers are closed under pullback and composition.*

Sheaf Conditions

Definition

Given $f : A \to X$ and $g : B \to X$ their **join** is the pushout

$$\begin{array}{ccc} A \times_X B & \longrightarrow & B \\ \downarrow & & \downarrow \\ A & \longrightarrow & A *_X B \end{array}$$

Given $f : A \to X$ write A_X^{*n} for the n-fold iterated join of f with itself.

Sheaf Conditions

Fix a presentation T.

Theorem (Sheaf Condition)

Let X be an n-type. Then X is a sheaf for T iff for all T-covers $f : A \to B$ the natural map

$$(B \to X) \to (A_B^{*n+2} \to X)$$

is an equivalence.

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Corollary

A 0-type X is a sheaf for T iff for all T-covers $f : A \to B$ the natural map

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In TTT: The type $\mathbb{I}$ is a T_{simp} sheaf.

Proof.

Using the sheaf conditions + the axioms of TTT $\mathbb{I}$ is a sheaf iff

$$\mathbb{I} \simeq \lim(\mathbb{I}/(i \leq j) \times \mathbb{I}/(j \leq i) \rightrightarrows \mathbb{I}/(i = j))$$

Pure algebra! □

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Example

Both of the running examples in SAG / TTT are projective (at least in models).

Lemma (T-local choice)

Let T be a projective presentation and C be projective. Let $B : C \to \mathcal{U}$ be such that $\prod_{c:C} \bigcirc_T \|B(c)\|$. Then there is a T-cover $f : Z \to C$ such that

$$\prod_{z:Z} B(f(z))$$

Cohomology

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Given any projective type X and any group G, $H^n(X, G) = 0$.

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How does cohomology descend to modalities from projective presentations?

Cohomology

Definition

Given a type $X : \mathcal{U}_T$ and a local group $G : \mathcal{U}_T$, its **cohomology** is given by

$$H^n_T(X, G) :\equiv \bigcirc_T \| X \to \bigcirc_T K(G, n) \|_0$$

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$$d^1(f)(x, x', x'') := f(x, x') - f(x, x'') + f(x', x'')$$

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Given an abelian group A and type X:

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Given a presentation T we will say A satisfies **descent** for T if for all $X \in T$ the above sequence is exact.

Theorem

Let T be a projective presentation and A an abelian group sheaf satisfying descent for T. Then for all projective X, we have $H^1_T(X, A) = 0$, where 0 is the trivial abelian group.

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Example (for algebraic geometry enthusiasts)

In SAG, an important class of modules (*quasi-coherent*) satisfy descent for T_{fppf}. Hence these have 0 cohomology on projectives - including R (and all *affine schemes*).

Thank you!

References

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