# PROPOSITIONAL GEOMETRIC TYPE THEORY HOTT/UF 2025

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# GEOMETRIC TYPE THEORY Setting & Idea

Kind of Space	Geometric Int. Lang	Full Int. Lang
Top <sub>sob</sub>	prop. geom. logic	compl. Heyting alg.
Loc	prop. geom. logic	compl. Heyting alg.
Topos	geom. logic	MLTT
$\infty$ -Topos	$\infty$ -geom. logic?	HoTT

► Toposes classify geometric theories T:

 $\operatorname{Topos}(\mathcal{E}, [\mathbb{T}]) \cong \operatorname{Mod}_{\mathbb{T}}(\mathcal{E}).$ 

- Geometric logic is incomplete, so we need to study models in all toposes.
- Some topos-valid constructions, such as Π-types, are not geometric/continuous, i.e. not preserved by inverse image functors.
- There are still toposes classifying arbitrary objects or maps, so geometric reasoning should suffice.
- ► This suggests treating toposes as types (cf. [Vic07]).

# GEOMETRIC TYPE THEORY MOTIVATION

- Recognition of geometric statements
- ► Transfer of results: *geometric* consequences of non-geometric statements are preserved
- Unification of external and internal perspective
- Unification of synthetic mathematics (SAG, SDG, STC)
- General treatment of modalities
- Recognition of classified geometric theories
- Synthetic Morita equivalences/bridges
- Definition of  $\infty$ -geometric logic
- Formalisation

# GEOMETRIC TYPE THEORY Related Work

There are lots of related ideas. None of them talk faithfully about *all* toposes, use the universal property of toposes as classifying spaces, and are an extension of HoTT.

- ► Topos-theoretic Multiverse [Ble]
- Multimodal Adjoint Type Theory [Shu23]
- Continuous Truth [Fou13]
- Abstract Stone Duality [Tay11]
- Synthetic Topology [Esc04]
- Synthetic Topos Theory [Uem]
- Arithmetic Type Theory [Vic08]

## GEOMETRIC TYPE THEORY Sketch of Intended Semantics

GTT should<sup>3</sup> be modeled by

 $Sh_{\infty}(Topos^{1}_{(2,1)}, J_{\acute{e}tale})$ 

with base type  $\mathbf{O} = [FinSet, Set]$  classifying étale spaces (a.k.a. internal types)

$$\begin{array}{ccc} \mathbf{T}(X) & \longrightarrow & \dot{\mathbf{O}} \\ \downarrow & {}^{-} & \downarrow \\ \mathcal{E} & \xrightarrow{X} & \mathbf{O} \end{array} \end{array},$$

letting us recover the Sierpiński space

$$\mathbf{S} = \sum_{X:\mathbf{O}} \operatorname{isProp}(\mathbf{T}(X))$$

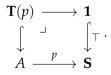
<sup>&</sup>lt;sup>3</sup>up to strictness and size issues, and a good choice of Grothendieck topology

# PROPOSITIONAL GEOMETRIC TYPE THEORY INTENDED SEMANTICS

Approximate above setting using semantics in the category

 $Sh_0(Loc,J_{opencover})$ 

of sheaves of sets on the category of localic toposes with the subcanonical open cover topology.
The Sierpiński space classifies open subtypes:



# PROPOSITIONAL GEOMETRIC TYPE THEORY SYNTAX

- Intensional MLTT
- Tarski-style universe

$$\frac{\Gamma \vdash p : \mathbf{S}}{\Gamma \vdash \mathbf{T}(p) \text{ type}}$$

- ▶ Bottom and top elements  $\bot, \top$  : S with  $T(\bot) \equiv 0$  and  $T(\top) \equiv 1$ .
- ▶ Primitive binary conjunctions  $\land : S \rightarrow S \rightarrow S$
- Order relation  $p \le q := (p \land q =_{\mathbf{S}} p)$
- Meet-semilattice axioms

## PROPOSITIONAL GEOMETRIC TYPE THEORY OVERT DISCRETE SPACES

•  $f : A \rightarrow B$  is open if

$$f^*: (B \to \mathbf{S}) \to (A \to \mathbf{S})$$

has a left adjoint  $f_!$ .

• *I* is overt if  $!: I \rightarrow \mathbf{1}$  is open, yielding

$$\bigvee: (I \to \mathbf{S}) \to \mathbf{S}.$$

- A is *discrete* if  $\Delta : A \to A \times A$  is open
- ► Assume **N** is overt discrete, **S**, **T**(*p*) overt.
- Being overt discrete is closed under positive type formers.

$$(\mathbf{T}(p) \to \mathbf{S}) \to \sum_{q:\mathbf{S}} (q \le p)$$
$$\varphi \mapsto p \land \bigvee_{x:\mathbf{T}(p)} \varphi(x)$$

is an equivalence.

# PROPOSITIONAL GEOMETRIC TYPE THEORY Directed Univalence

For *A* and *B* overt discrete we should have *directed univalence* 

$$(A \to B) \tilde{\to} \sum_{\gamma: \mathbf{S} \to \text{ODisc}} (\gamma(\bot) = A) \times (\gamma(\top) = B)$$
$$f \mapsto \lambda p. \sum_{b: B} \mathbf{T}(p) \star \text{fib}_f(b),$$

just like in Condensed Type Theory [Bar24, Com24].

# PROPOSITIONAL GEOMETRIC TYPE THEORY CURRENT WORK

- Characterise the topology of function spaces
- Use synthetic quasi-coherence and local choice
- ► Justify usability of our theory by proving (cf. [Hyl81])

$$\big((N \to 2) \to 2\big) \simeq N$$

- Extend simplicial aspects
- Extend to full Geometric Type Theory

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