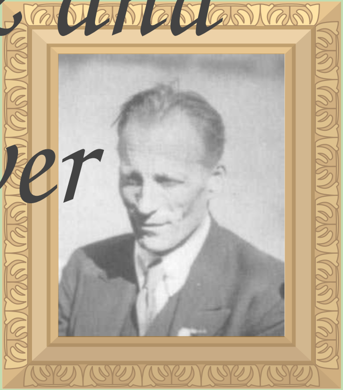
Hurewicz and Brouwer



Axel Ljungström, Loïc Pujet

16 april 2025

Disclaimer

This presentation is not about the history of mathematics.
More honest title:

The cellular Hurewicz theorem in constructive HoTT

The Cellular Hurewicz Theorem

Let X be a $(n-1)$-connected CW complex.

Then $\pi_n^{ab}(X) \cong H_n^{cw}(X)$.

The Cellular Hurewicz Theorem

$\|X\|_{n-1} \simeq 1$

type built out of spherical cells

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abelianised homotopy group

cellular homology group

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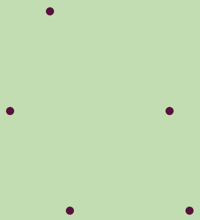

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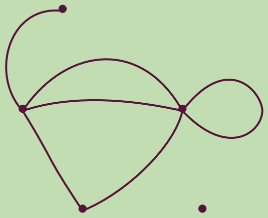

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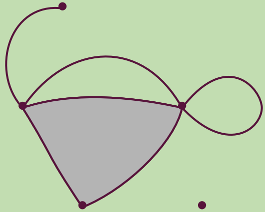

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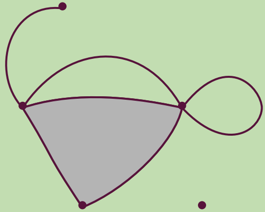

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Caveat: to work with CW complexes, we often need the axiom of choice for type families indexed over the sets of cells

→ not a problem if we only allow finite sets of cells
(more generally, we can allow projective sets of cells)

Cellular Homology

We can define cellular homology groups for CW complexes, following the traditional definition from algebraic topology

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- measure the number of n-dimensional holes of X
- independent of the cellular structure (homotopy invariant)
- satisfy the Eilenberg Steenrod axioms

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And yet, related by Hurewicz's theorem:

If X is $(n-1)$-connected, then $\pi_n^{ab}(X) \cong H_n^{cw}(X)$

Hurewicz connectedness

Say that a CW complex is Hurewicz n-connected when it has

- exactly one 0-cell
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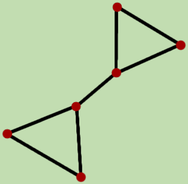

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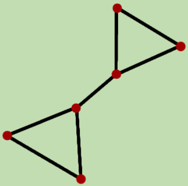

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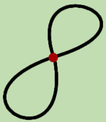

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Theorem

If X is Hurewicz $(n-1)$-connected, then $\pi_n^{ab}(X) \cong H_n^{cw}(X)$

Proof

Follows from Blakers–Massey and excision □

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Classical strategy: replace the set of $(n+1)$-cells with a generating set for $\pi_{n+1}(X)$

→ We need to show that $\pi_{n+1}(X)$ is finitely (projectively) generated

→ Not true for finite CW complexes: $\pi_2(\mathbb{S}^1 \vee \mathbb{S}^2) \simeq \bigoplus_{\mathbb{Z}} \mathbb{Z}$

Second attempt

Dimension 0

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Proceed by induction on the number of vertices

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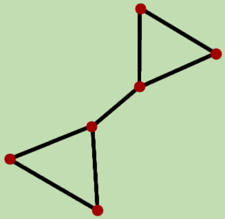

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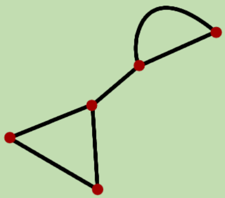

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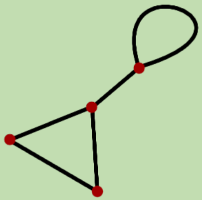

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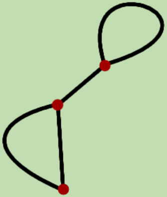

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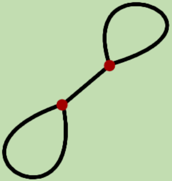

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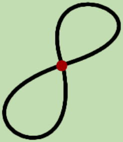

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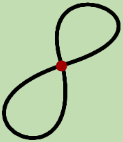

→ Only works for finite sets of cells
→ unclear how to generalise this reasoning to dimensions > 0

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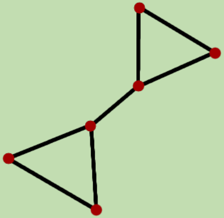

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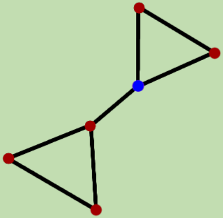

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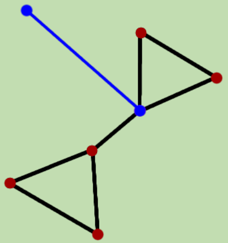

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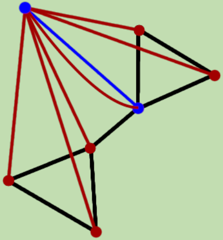

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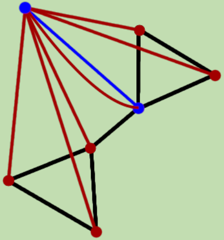

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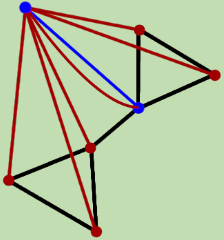

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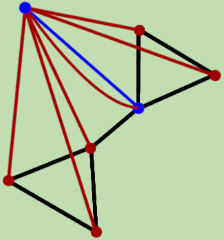

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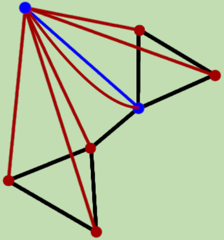

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Conclude by adding c_0 2-cells.

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We conclude by adding c_n $(n+2)$-cells.

Conclusion

Theorem (constructively)

Every n-connected CW complex is equivalent to a Hurewicz n-connected CW complex

Theorem (constructively)

If X is $(n-1)$-connected, then $\pi_n^{ab}(X) \cong H_n^{cw}(X)$

Applications

Serre Finiteness Theorem (Barton and Campion '22, Milner '23)

Thank you!