Transpension
for Cubes without Diagonals

Andreas Nuyts

DistriNet KU LEUVEN fwo

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https://anuyts.github.io/#trascwod

Presheaf semantics can model:

- **HoTT** (preservation of **isomorphisms**),
- **Parametricity** (preservation of **relations**),
- **Guarded TT** (preservation of **stage of computation**),
- **Nominal TT** (preservation of **renaming** and α**-equivalence**),
- **Directed TT** (preservation of **homomorphisms**).

Use these preservation properties **within type theory**?

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Internalization Operators

We want **simpler foundations:**

- simpler metatheory,
- cross-fertilization (e.g. affine Gel vs. cartesian $\surd$),
- guidance (e.g. directed TT).

	Transpension $\forall(u : \mathbb{U}) \dashv \between[u]$ (Yet87, ND22)		**Strictness** (OP18)	**Pushout**
HoTT $(\mathbb{I} \to \sqcup) \dashv \surd$ (LOPS18)			HoTT/DirTT/Param. Glue (CCHM16/ NVD17/WL20)	Param. Weld (NVD17)
Nominal TT $Иi, \langle\langle i \rangle\rangle, \nu\, i$ (PMD14)	Func. param. Φ/extent (BCM15)	Type param. Ψ/Gel (BCM15)	mill Weld $\leftrightarrow \forall\, i$ (ND18)	Guarded TT Löb ind.

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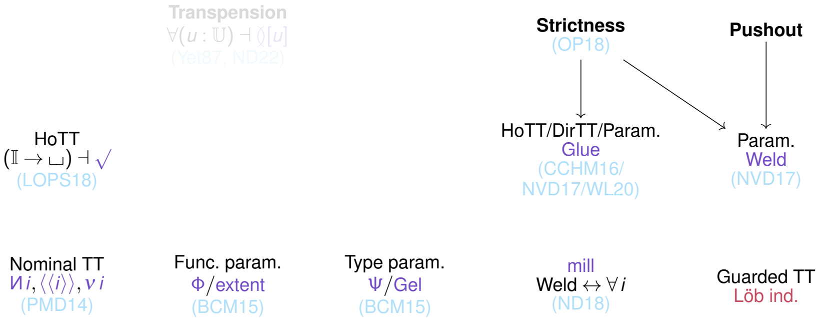

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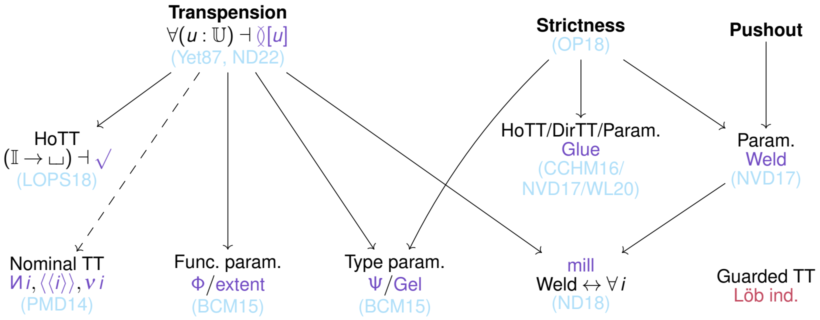

What is this **Transpension Type**?

$$\forall u : \mathrm{Ty}(\Gamma, u : \mathbb{U}) \to \mathrm{Ty}(\Gamma) \quad\dashv\quad ()[u] : \mathrm{Ty}(\Gamma) \to \mathrm{Ty}(\Gamma, u : \mathbb{U})$$

Let's look at it for **affine cubes**, as are used

- for HoTT [BCH14]
- for parametricity [BCM15, CH21]

We purely use **adjointness**:

$$\frac{\Gamma \vdash \_ : (\forall i.S[i]) \to T}{\Gamma, i : \mathbb{I} \vdash \_ : S[i] \to ()[i]\, T} \qquad \frac{\Gamma, (\forall i.\Delta[i]) \vdash \_ : T}{\Gamma, i : \mathbb{I}, \Delta[i] \vdash \_ : ()[i]\, T}$$

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Poles

$$\dfrac{\dfrac{\dfrac{\Gamma \vdash \_ : \between[0]\, A}{\Gamma, i : \mathbb{I}, \_ : (i = 0) \vdash \_ : \between[i]\, A}}{\Gamma, \_ : \forall i.(i = 0) \vdash \_ : A}}{\Gamma, \_ : \bot \vdash \_ : A}$$

(each step is a double-line rule)

- So $\between[0]\, A$ and $\between[1]\, A$ are **uniquely inhabited** by pole.

Meridians

$$\dfrac{\dfrac{\Gamma \vdash \_ : \forall i.\between[i]\, A}{\Gamma, i : \mathbb{I}, () \vdash \_ : \between[i]\, A}}{\Gamma, \forall i.() \vdash \_ : A}$$

(each step is a double-line rule)

- **Sections** of $\between[i]\, A$
  are called **meridians**
  as they connect the poles,
  and correspond to **elements of** A.

Transpension ≈ dependent **suspension**

Poles

$$\Gamma \vdash \_ : \between[0]\, A$$
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\hline\hline
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Higher-dimensional pattern matching

$$
\begin{array}{lll}
\Gamma & \vdash & \forall i.(A_1\, i \uplus A_2\, i) \to (\forall i.A_1\, i) \uplus (\forall i.A_2\, i) \\
\hline\hline
\Gamma, i : \mathbb{I} & \vdash & (A_1\, i \uplus A_2\, i) \to \between[i]\,((\forall i.A_1\, i) \uplus (\forall i.A_2\, i)) \\
\hline\hline
\Gamma, i : \mathbb{I} & \vdash_{i=1,2} & A_i\, i \to \between[i]\,((\forall i.A_1\, i) \uplus (\forall i.A_2\, i)) \\
\hline\hline
\Gamma & \vdash_{i=1,2} & \mathsf{inj}_i : \forall i.A_i\, i \to (\forall i.A_1\, i) \uplus (\forall i.A_2\, i)
\end{array}
$$

Presheaf Semantics of **Transpension**

Transpension

$$\forall u \quad : \quad \mathrm{Ty}(\Xi, u : \mathbb{U}) \to \mathrm{Ty}(\Xi) \quad \dashv$$
$$\between[u] \quad : \quad \mathrm{Ty}(\Xi) \to \mathrm{Ty}(\Xi, u : \mathbb{U})$$

$$\forall_{\overline{U}} \quad : \quad \mathrm{Psh}(\textstyle\int_{\mathscr{W}}(\Xi, u : \mathbb{U})) \to \mathrm{Psh}(\textstyle\int_{\mathscr{W}} \Xi) \quad \dashv$$
$$\between_{\overline{U}} \quad : \quad \mathrm{Psh}(\textstyle\int_{\mathscr{W}} \Xi) \to \mathrm{Psh}(\textstyle\int_{\mathscr{W}}(\Xi, u : \mathbb{U}))$$

Presheaf semantics in a context

Working over $\int_{\mathscr{W}} \Xi \quad \sim \quad$ working in context Ξ.

TT in $\mathrm{Psh}(\int_{\mathscr{W}} \Xi)$		TT in $\mathrm{Psh}(\mathscr{W})$
		Ξ ctx
Γ ctx	$\sim$	$\Xi.\Gamma$ ctx
$\Gamma \vdash T$ type	$\sim$	$\Xi.\Gamma \vdash T$ type
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$$\frac{\text{TT in } \mathrm{Psh}(\int_{\mathscr{W}} \Xi) \qquad \text{TT in } \mathrm{Psh}(\mathscr{W})}{}$$

		$\Xi\ \mathsf{ctx}$
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$\Gamma \vdash T\ \mathsf{type}$	$\sim$	$\Xi.\Gamma \vdash T\ \mathsf{type}$
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$$\mathscr{W} \times \mathscr{W} \xrightarrow{\otimes} \mathscr{W}$$

$$\mathrm{Psh}(\mathscr{W}) \times \mathrm{Psh}(\mathscr{W}) \qquad \mathrm{Psh}(\mathscr{W})$$

Precise **semantics?**

- **Monoidal** base category $(\mathscr{W}, E, \otimes)$
- **Day convolution** $(\mathrm{Psh}(\mathscr{W}), \mathbf{y}E, \widehat{\otimes})$
  $\mathbf{y}(W \otimes U) \cong \mathbf{y}W \,\widehat{\otimes}\, \mathbf{y}U$
- Choose a **base object** ("shape") U.

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- $(- \otimes U)$ lifts to **elements**:
  Let $\Xi \in \mathrm{Psh}(\mathscr{W})$

$$\exists_U^{\int \Xi} : \int_{\mathscr{W}} \Xi \to \int_{\mathscr{W}} (\Xi \,\widehat{\otimes}\, \mathbf{y}U)$$
$$\exists_U^{\int \Xi}(W, \xi : \mathbf{y}W \to \Xi) :=$$
$$(W \otimes U, \xi \,\widehat{\otimes}\, \mathbf{y}U : \mathbf{y}(W \otimes U) \to \Xi \,\widehat{\otimes}\, \mathbf{y}U)$$

- **Assume** $\exists_U^{\int \Xi} \dashv \reflectbox{$\exists$}_U^{\int \Xi}$,
  (equiv.: $(- \otimes U)$ is a **param. r. adj.**)
  ✔ **True** in all applications of interest.

Transpension

$$\forall u : \mathrm{Ty}(\Xi, u : \mathbb{U}) \to \mathrm{Ty}(\Xi) \quad \dashv$$
$$\between[u] : \mathrm{Ty}(\Xi) \to \mathrm{Ty}(\Xi, u : \mathbb{U})$$

$$\forall_U^{\int \Xi} : \mathrm{Psh}(\textstyle\int_{\mathscr{W}}(\Xi, u : \mathbb{U})) \to \mathrm{Psh}(\int_{\mathscr{W}} \Xi) \quad \dashv$$
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Precise **semantics?**

- **Monoidal** base category $(\mathscr{W}, E, \otimes)$
- **Day convolution** $(\mathrm{Psh}(\mathscr{W}), \mathbf{y}E, \widehat{\otimes})$
  $\mathbf{y}(W \otimes U) \cong \mathbf{y}W \,\widehat{\otimes}\, \mathbf{y}U$
- Choose a **base object** ("shape") U.

- $(- \otimes U)$ lifts to **elements**:
  Let $\Xi \in \mathrm{Psh}(\mathscr{W})$

$$\exists_U^{\int \Xi} : \int_{\mathscr{W}} \Xi \to \int_{\mathscr{W}} (\Xi \,\widehat{\otimes}\, \mathbf{y}U)$$
$$\exists_U^{\int \Xi}(W, \xi : \mathbf{y}W \to \Xi) :=$$
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We have

$$\exists_U^{\int \Xi} \dashv \reflectbox{$\exists$}_U^{\int \Xi} : \int_{\mathscr{W}} \Xi \to \int_{\mathscr{W}} (\Xi \mathbin{\widehat{\otimes}} \mathbf{y} U),$$

whence 4 adjoint functors between $\mathrm{Psh}(\int_{\mathscr{W}} \Xi)$ and $\mathrm{Psh}(\int_{\mathscr{W}} (\Xi \mathbin{\widehat{\otimes}} \mathbf{y} U))$

$$\begin{array}{ccccccc}
(\exists_U^{\int \Xi})_! & \dashv & (\exists_U^{\int \Xi})^* & \dashv & (\exists_U^{\int \Xi})_* & & \\
 & & \| \wr & & \| \wr & & \\
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Given

- a **monoidal** base category $(\mathscr{W}, E, \otimes)$ with **object** U,

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$$\wr \qquad\qquad \wr$$

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Ok great, so

Give Me A Syntax!

Syntax

Problem

- Unit performs justified **variable capture:**
  $\Gamma, u : \mathbb{U} \vdash \eta : A[u] \to \between[u]\,(\forall(v : \mathbb{U}).A[v])$
- This **cannot** commute with **contraction:**
  $(u/w) : (u : \mathbb{U}) \to (w : \mathbb{U}, u : \mathbb{U})$

Solution: ~~contraction~~ → **affine/linear** shape variables.

Tradeoff between **generality** and **well-behavedness**:

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MTraS:
Modal Transpension System
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- Allows **contraction**,
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Paper [ND24] lists **criteria** for $(- \otimes U)$ that make **transpension better behaved**.

$$\overbrace{\Gamma,}^{\text{fresh for } u} \; u : \mathbb{U}, \; \overbrace{\Delta}^{\text{not fresh for } u} \; \text{ctx}$$

Example

If $\dashv^{\int\top}_U$ is **fully faithful** then:

(equiv.: $\dashv^{\int\Xi}_U$ is f.f. for all Ξ)

- $\between_U$ is also ff, so $\forall_U \circ \between_U \cong \mathrm{Id}$,
- and contraction is disallowed

FFTraS assumes this is the case.

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FF:TRANSP:INTRO

$$\frac{\Gamma, \forall u. \Delta \vdash a : A}{\Gamma, u : \mathbb{U}, \Delta \vdash \mathsf{mer}[u]\, a : \between[u]\, A}$$

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Problem: FFTraS is a naïve system.

Close this system under substitution $\sigma : \Theta \to (\Gamma, u : \mathbb{U}, \Delta)$

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TraSCwoD:

Transpension **S**ystem for **C**ubes **w**ith**o**ut **D**iagonals

TraSCwoD: Base Category

Shape calculus $\mathscr{S}$

Category $\mathscr{S}$ of **shapes** with **minimal object** $\diamond$ (i.e. no incoming arrows)

Idea: 0/1-variable relevant calculus with constants.

Cube category $\mathscr{C}$

Let $\mathscr{C} = \mathscr{S}^{\otimes}, \mathscr{S}^{\otimes}_{\sigma}, \mathscr{S}^{\otimes}_{\pi}, \mathscr{S}^{\otimes}_{\pi,\sigma}$ be the **free**

- σ **symmetric**
- π **monoidal**
- π **semicartesian** (i.e. terminal unit)

category **with unit** $\diamond$ over $\mathscr{S}$.

Example

Affine cubes	$\mathscr{C} = \mathscr{S}^{\otimes}_{\pi,\sigma}$	$\mathscr{S} = \{\diamond \rightrightarrows^0_1 \mathbb{I}\}$	[BCH14,BCM15]
Nominal TT		$\mathscr{S} = \{\diamond \quad \mathbb{I}\}$	
Affine DoR		$\mathscr{S} = \{\diamond \rightrightarrows^0_1 \mathbb{I}_{d-1} \to \ldots \to \mathbb{I}_1 \to \mathbb{I}_0\}$	[NVD17,ND18]
Affine clocks		$\mathscr{S} = \{\diamond \quad 🕒_0 \to 🕒_1 \to 🕒_2 \to \ldots\}$	

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TraSCwoD: Base Category

Shape calculus $\mathscr{S}$

Category $\mathscr{S}$ of **shapes** with
minimal object $\diamond$ (i.e. no incoming arrows)
Idea: 0/1-variable relevant calculus with constants.

Cube category $\mathscr{C}$

Let $\mathscr{C} = \mathscr{S}^{\otimes}, \mathscr{S}^{\otimes}_{\sigma}, \mathscr{S}^{\otimes}_{\pi}, \mathscr{S}^{\otimes}_{\pi,\sigma}$ be the **free**

- σ **symmetric**
- $\not\pi$ **monoidal**
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TraSCwoD: Typing rules: $\forall$

Uncontroversial rules (both FFTraS & TraSCwoD):

CWOD:FORALL

$$\frac{\Gamma, u : \mathbb{U} \vdash A \,\mathrm{type}}{\Gamma \vdash \forall u.A \,\mathrm{type}}$$

CWOD:FORALL:INTRO

$$\frac{\Gamma, u : \mathbb{U} \vdash a : A}{\Gamma \vdash \lambda u.a : \forall u.A}$$

What about **elimination?**

- $\forall u$ is a **DRA**[BCMMPS20] to $(-, u : \mathbb{U})$
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Typing rule for **DRA** to **PRA** [GCKGB22]:

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$$\frac{\Theta \vdash r : \mathbb{U} \qquad \exists_{\mathbb{U}}(\Theta, r) \vdash f : \forall u.A}{\Theta \vdash f\, r : A[r/u]}$$

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Adapt **FFTraS** ➔ **TraSCwoD**:

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Elimination: unmer : $(\forall u.\between[u] A) \to A$

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TraSCwoD: Funny Contexts

What are $\exists_{\mathbb{U}}(\Theta, r)$ and $\forall_{\mathbb{U}}(\Theta, r)$?

- **Decreed** context constructors do **not tell you** how to use **variables**.
- **Recursively** defined context operations have **no semantic** counterpart.
- Have a **recursive best approximation** justified syntactically from $\exists_{\mathbb{U}} \dashv \exists_{\mathbb{U}} \dashv \forall_{\mathbb{U}}$:

$$\exists_{\mathbb{U}}(\Theta, r) \to \Theta/r, \qquad \forall_{\mathbb{U}}(\Theta, r) \to \Theta^r.$$

For **affine cubes**:

$$\begin{array}{lllllllll} \Theta/0 & = & \Theta & \Theta/1 & = & \Theta & (\Gamma, i : \mathbb{I}, \Delta)/i & = & \Gamma \\ \Theta^0 & = & \bot & \Theta^1 & = & \bot & (\Gamma, i : \mathbb{I}, \Delta)^i & = & (\Gamma, \forall i.\Delta) \end{array}$$

WLOG no shape variables in Δ.

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Conclusion

By limiting **base categories** to **linear/affine cube categories**,
we can get a **well-behaved type system** with **transpension**!

Thanks!

Questions?

MTraS: Modal Transpension System [ND24]

MTraS allows **contraction**

→ **Problems** with shape substitution.

Solution:

- Instantiate **MTT** (Multimod[e/al] Type Theory) [GKNB21],
- Put **shapes** & **(co)quantifiers** in the **mode theory**.

Tradeoff:

- ☺ It **exists!**
- ☺ It is **sound!**
- ☺ As **general** as the semantics!
- ☺ We can **study** and **use** the transpension type.
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Wanted: something

☺ **simpler**,

☹ **less general**,

☺ still covering **interesting applications**.