Châtelet's Theorem in Synthetic Algebraic Geometry

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Overview

Algebraic geometry is the study of roots of systems of polynomials.

Theorem [Châtelet 1944]

Any inhabited Severi-Brauer variety is a projective space.

Goal

Give a synthetic proof of Châtelet's theorem.

Overview

Today

1. Introduce synthetic algebraic geometry.
2. Define étale sheaves. Being an étale sheaf is a lex modality.
3. Use étale sheafification to define Severi-Brauer varieties.
4. Sketch the proof of Châtelet's theorem.
   Essentially lex modality reasoning + some linear algebra.

Main point of this talk

Working with étale sheaves in synthetic algebraic geometry is convenient and helpful.

Outline

What is synthetic algebraic geometry?

It consists of HoTT plus 3 axioms:

Axiom 1

There is a local ring R.

R is assumed to be a set.

Affine schemes

For A a finitely presented algebra, we define:

$$Spec(A) = Hom_{R\text{-}Alg}(A, R)$$

Example

If:

$$A = R[X]/P$$

then:

$$Spec(A) = \{x : R \mid P(x) = 0\}$$

Definition

A type X is an affine scheme if there is an f.p. algebra A such that:

$$X = Spec(A)$$

Axiom 2: Duality

For any f.p. algebra A the map:

$$A \to R^{Spec(A)}$$

is an equivalence.

Then:

- $Spec : \{f.p.\ algebras\} \simeq \{Affine\ schemes\}$
- All maps between affine schemes are polynomials.

Axiom 3: Zariski local choice

Affine schemes enjoys a weakening of the axiom of choice.

Schemes

Defintion

A type is a scheme if it has a finite open cover by affine schemes.

Example

The projective space:

$$\mathbb{P}^n = \{Lines\ in\ R^{n+1}\ going\ through\ the\ origin\}$$

is a scheme.

Châtelet's theorem in traditional algebraic geometry

Definition (Traditional algebraic geometry, k a field)

A Severi-Brauer variety over k is a k-scheme X such that X is a projective space over the separable closure of k.

Theorem [Châtelet 1944]

Any inhabited Severi-Brauer variety is a projective space.

Synthetically, for X a Severi-Brauer variety:

$$\|X\| \to \exists(n : \mathbb{N}).\, \|X = \mathbb{P}^n\|$$

Challenge

How to define synthetic Severi-Brauer varieties?

Outline

Definition

Definition

A type X is an étale sheaf if for any monic unramifiable $P : R[X]$, we have a unique filler:

$$\begin{array}{ccc} \exists(x : R).\, P(x) = 0 & \longrightarrow & X \\ \downarrow & \overset{\exists!}{\nearrow} & \\ 1 & & \end{array}$$

Étale sheaves behave as if any monic unramifiable P had a root.

Remark

By [Wraith 79], this should correspond to traditional étale sheaves.

Being an étale sheaf is a lex modality.
So we have an étale sheafification [Rijke, Shulman, Spitters 2017].

Remark

HoTT can be interpreted inside étale sheaves:

- The universe U is interpreted as the type U_{Et} of étale sheaves.
- Σ, Π and identity types are interpreted as themselves.
- Truncation $\|\_\|$ is interpreted as the étale sheafification of the truncation, denoted $\|\_\|_{Et}$.

Required properties of étale sheaves

Lemma

Any scheme is an étale sheaf.

Lemma

Let M be a module that is an étale sheaf.
The proposition "M is finite free" is an étale sheaf.

Equivalently, the type of finite free module is an étale sheaf.

Remark

Traditionally phrased as étale descent for finite free modules.

Outline

Introduction to synthetic algebraic geometry

Étale sheaves

Severi-Brauer varieties and Châtelet's theorem

Severi-Brauer varieties

We fix a natural number n.

Definition

A Severi-Brauer variety is an étale sheaf X such that $\|X = \mathbb{P}^n\|_{Et}$.

We write:

$$SB_n = \{X : U_{Et} \mid \|X = \mathbb{P}^n\|_{Et}\}$$

Example: Conics

Assume $2 \neq 0$.

Example

For all $a, b : R$ invertible:

$$C(a, b) = \{[x : y : z] : \mathbb{P}^2 \mid ax^2 + by^2 = z^2\}$$

is a Severi-Brauer variety.

Severi-Brauer varieties form a delooping

Theorem [Cherubini, Coquand, Hutzler, Wärn 2024]

$$Aut(\mathbb{P}^n) = PGL_{n+1}(R)$$

Therefore:

$$SB_n = \{X : U_{Et} \mid \|X = \mathbb{P}^n\|_{Et}\}$$

is the interpretation inside étale sheaves of:

$$\{X : U \mid \|X = \mathbb{P}^n\|\} = B(Aut(\mathbb{P}^n)) = B(PGL_{n+1}(R))$$

Azumaya algebras form a delooping

Lemma

$$Aut_{R\text{-}Alg}(M_{n+1}(R)) = PGL_{n+1}(R)$$

So we define the type of Azumaya algebras:

$$AZ_n = \{A : R\text{-}Alg_{Et} \mid \|A = M_{n+1}(R)\|_{Et}\}$$

It is also the interpretation of $B(PGL_{n+1}(R))$ inside étale sheaves.

Severi-Brauer varieties from Azumaya algebras

By unicity of deloopings, we know that $AZ_n \simeq SB_n$.
We can be more concrete:

Lemma

The pointed map:

$$\begin{aligned} LI &: AZ_n \to SB_n \\ LI(A) &= \{I \text{ left ideals in } A \mid I \text{ free of rank } n+1\} \end{aligned}$$

is an equivalence.

Proof sketch

LI induces a map:

$$\alpha : PGL_{n+1}(R) = \Omega AZ_n \xrightarrow{\Omega LI} \Omega SB_n = PGL_{n+1}(R)$$

By lex modality reasoning, we just need that α is the identity.
Check this by direct computation.

Lemma

Given $A : AZ_n$, if we have $I : LI(A)$ then $A = End_R(I)$.

Proof sketch

By lex modality reasoning, we can assume $A = M_{n+1}(R)$.
Then we do some linear algebra.

Corollary

Given $A : AZ_n$, if we have $\|LI(A)\|$ then $\|A = M_{n+1}(R)\|$.

Châtelet's theorem in synthetic algebraic geometry

Theorem

For all $X : SB_n$, we have that $\|X\|$ implies $\|X = \mathbb{P}^n\|$.

Proof

$$\begin{array}{rcl} & \|X\| & \\ \Rightarrow & \|LI(A)\| & (\textit{write } A = LI^{-1}(X)) \\ \Rightarrow & \|A = M_{n+1}(R)\| & (\textit{previous Corollary}) \\ \Rightarrow & \|X = LI(M_{n+1}(R))\| & (\textit{apply } LI) \\ \Rightarrow & \|X = \mathbb{P}^n\| & (LI \textit{ pointed}) \end{array}$$