Some properties of Whitehead products

Axel Ljungström

HoTT/UF, 16 April, 2025

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~~Genoa, Italy~~

Stockolm, Sweden

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- This talk is about the *Whitehead product*: a graded multiplication

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- These are useful for producing non-trivial elements of homotopy groups
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- Classically, this bracket has some nice Lie algebra properties:
  - Bilinearity: $[f+g,h] = [f,h]+[g,h]$ and $[f,g+h] = [f,g]+[f,h]$
  - Graded symmetry $[f,g] = (-1)^{nm}[g,f]$
  - Graded Jacobi identity: $(-1)^{nk}[f,[g,h]] + (-1)^{nm}[g,[h,f]] + (-1)^{km}[h,[f,g]] = 0$

Above, $n = \deg(f)$*,* $m = \deg(g)$ *and* $k = \deg(h)$

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A brief history of Whitehead products in HoTT

- 2016: Brunerie defines gives the a HoTT definition of Whitehead products in his PhD thesis.
- 2019: Ali Caglayan writes a nice post about Whitehead products as commutators on the HoTT mailing list.
- 2023: Buchholtz et. al (in *Central H-spaces and banded types*) relate, among other things, the vanishing of certain Whitehead products to H-space structures.
- 2024: Cagne et. al (in *On symmetries of spheres in univalent foundations*) construct and EHP sequence and compute $\pi_1(\mathbb{S}^2 \to \mathbb{S}^2)$, **assuming bilinearity** of Whitehead products.
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- Conclusion: there is some interest in Whitehead products in HoTT *but we still don't know much about their basic properties*
- In particular bilinearity seems important. This was my original motivation.
  - Proof technique can also be used for symmetry and Jacobi – a lucky coincidence!

Some light background

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- We define the **smash product** of two pointed types X and Y, denoted $X \wedge Y$, to be the HIT generated by
  - points $\langle x, y \rangle : X \wedge Y$ for $x : X$ and $y : Y$,
  - an additional basepoint and higher constructors forcing $X \wedge Y$ to be a 'tensor product'

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- The suspension a type X, denoted ΣX can be defined e.g. by $\Sigma X := \mathbb{S}^1 \wedge X$...
- ... but we don't need implementation details here. All we need to know is that when X is pointed, there is an adjunction $\Sigma \dashv \Omega$:

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- We define 'addition' of two functions $f, g : \Sigma X \to_\star Y$ by defining $\widetilde{f + g} : X \to_\star \Omega Y$ by

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- In particular, this gives the group structure on homotopy groups, $\pi_n(X) := \| \mathbb{S}^n \to_\star X \|_0$
  - Recall, $\mathbb{S}^{n+1} = \Sigma \mathbb{S}^n$

Defining Whitehead products

Definition 1

Let X and Y be pointed types. Given $f : \Sigma X \to_\star Z$ and $g : \Sigma Y \to_\star Z$, we define their generalised Whitehead product (GWP)

$$[f, g] : \Sigma(X \wedge Y) \to_\star Z$$

using the $(\Sigma \dashv \Omega)$-adjunction. We define $\widetilde{[f, g]} : X \wedge Y \to_\star \Omega Z$ (on points) by

$$\widetilde{[f, g]}\langle x, y\rangle = \widetilde{f}(x)^{-1}\widetilde{g}(y)\widetilde{f}(x)\widetilde{g}(y)^{-1}$$

Disclaimer

The GWP is often defined with the join $X * Y$ as domain. We have $\Sigma(X \wedge Y) \simeq X * Y$, so this is equivalent.

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- It gives rise to the standard one on homotopy groups by setting $X = \mathbb{S}^n$ and $Y = \mathbb{S}^m$. Using that spheres are suspensions and that $\mathbb{S}^n \wedge \mathbb{S}^m \simeq \mathbb{S}^{n+m}$, we get something of the right form.

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- Idea: instead of proving properties for the standard Whitehead product, prove their generalised analogues.

Bilinearity

- The original statement of **bilinearity** is still well-typed for the GWP.
- For instance, left-linearity says that for $f, g : \Sigma X \to_\star Z$ and $h : \Sigma Y \to_\star Z$, we have that $[f + g, h] = [f, h] + [g, h]$.

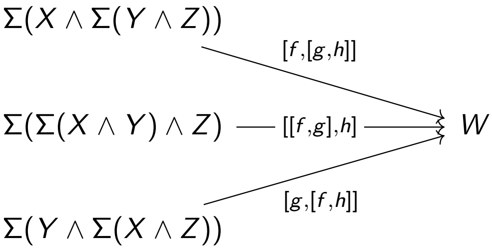
Symmetry

- **Symmetry** can be interpreted as saying that the following diagram commutes for $f : \Sigma X \to_\star Z$ and $g : \Sigma Y \to_\star Z$.

$$
\begin{array}{ccc}
\Sigma(X \wedge Y) & \xrightarrow{[f,g]} & Z \\
\downarrow & & \uparrow{\scriptstyle [g,f]} \\
\Sigma(Y \wedge X) & \xrightarrow[(-)^{-1}]{} & \Sigma(Y \wedge X)
\end{array}
$$

Jacobi

- The **Jacobi identity** asks us to identify three maps which appear to have different domain.



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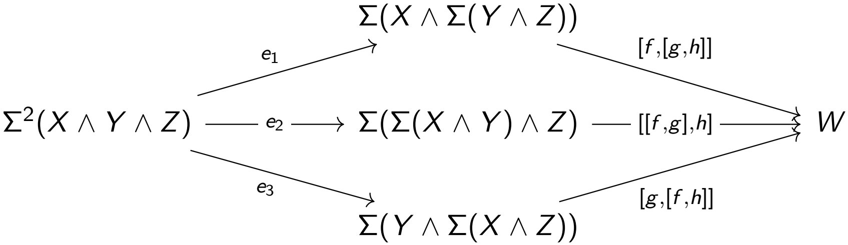

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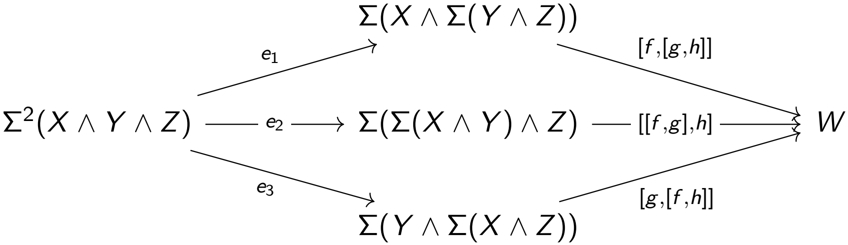

- The identity can then be stated as

$$[f,[g,h]] \circ e_1 = [[f,g],h] \circ e_2 + [g,[f,h]] \circ e_3$$

Towards proving the properties

- Recall, we never constructed the GWP $[f, g] : \Sigma(X \wedge Y) \to_\star Z$ directly. Instead, we constructed $\widetilde{[f, g]} : X \wedge Y \to_\star \Omega Z$
- we will keep working of this side of the $(\Sigma \dashv \Omega)$-adjunction when we prove the properties
- Let's translate them over to this side of the adjunction.

- In what follows, let $f : \Sigma X \to_\star W$, $g : \Sigma Y \to_\star W$ and $h : \Sigma Z \to_\star W$. Assume that X, Y and Z are themselves suspensions (e.g. $X = \Sigma X'$ for some pointed X').
- Fix $(x, y, z) : X \times Y \times Z$. For ease of notation, let us simply write $\mathbf{x}, \mathbf{y}, \mathbf{z} : \Omega W$ for, respectively, $\widetilde{f}(x), \widetilde{g}(y), \widetilde{h}(z)$. We have:

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- **Left-linearity**:

$$\underbrace{\mathbf{z}^{-1}\mathbf{x}\mathbf{y}\mathbf{z}\mathbf{y}^{-1}\mathbf{x}^{-1}}_{[f+g,h]} = \underbrace{\mathbf{z}^{-1}\mathbf{x}\mathbf{z}\mathbf{x}^{-1}\mathbf{z}^{-1}\mathbf{y}\mathbf{z}\mathbf{y}^{-1}}_{[f,h]+[g,h]}$$

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- **Jacobi**:

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- **Symmetry**:

$$\mathbf{x}^{-1}\mathbf{y}\mathbf{x}\mathbf{y}^{-1} = \mathbf{x}\mathbf{y}^{-1}\mathbf{x}^{-1}\mathbf{y} \qquad \leftarrow \text{Let's check the easiest one!}$$

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Proving symmetry: $\mathbf{x}^{-1}\mathbf{y}\mathbf{x}\mathbf{y}^{-1} = \mathbf{x}\mathbf{y}^{-1}\mathbf{x}^{-1}\mathbf{y}$

- Proof idea: just commute paths...

$$\underbrace{\mathbf{x}^{-1}\mathbf{y}}\,\underbrace{\mathbf{x}\mathbf{y}^{-1}} = \mathbf{x}\mathbf{y}^{-1}\mathbf{x}^{-1}\mathbf{y}$$

- ...but this move is (a priori) illegal.
- *However*, thanks to the additional suspension assumption, we are allowed so commute some paths...

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- The LHS consist of two words which we view as functions of type $Y \to_\star \Omega W$:

$$w_1(y) := \mathbf{x}^{-1}\mathbf{y}\mathbf{x} \quad w_2(y) := \mathbf{y}^{-1}$$

- Because $Y = \Sigma Y'$ and both functions are pointed, it is an easy consequence of Eckmann-Hilton that $w_1(y)w_2(y) = w_2(y)w_1(y)$.

Proving symmetry: $x^{-1}yxy^{-1} = xy^{-1}x^{-1}y$

- So we can rewrite the LHS:

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- We now play the same game, viewing the RHS above as a composite of two words $w_3, w_4 : X \to_\star \Omega W$
- These commute for the same reason as before. So

$$w_3(x)w_4(x) = w_4(x)w_3(x) = \mathbf{x}\mathbf{y}^{-1}\mathbf{x}^{-1}\mathbf{y}$$

and we are done!

Wrapping up

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  - A bit more complicated but certainly solvable ‘word problems’
- When translating back to the standard Whitehead product, the suspension assumption simply corresponds to the fact that the properties hold for homotopy groups in dimension > 1.
- Main takeaways:
  - Whitehead products behave as expected in HoTT. This is a good thing.
  - The proof I sketched here reduces the properties of Whitehead products to easy ‘word problems’ and thereby simplifies the classical proofs (that I’m aware of) quite substantially.

Thanks for listening!

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