Extensional concepts in intensional type theory, revisited

References

Background

Hofmann, Martin. Extensional constructs in intensional type theory. PhD thesis, 1995.

Kapulkin, Krzysztof and Lumsdaine, Peter LeFanu. The homotopy theory of type theories. Advances in Mathematics, 2018.

Isaev, Valery. Morita equivalences between algebraic dependent type theories. arXiv:1804.05045, 2020.

Main result

Kapulkin, Krzysztof and Li, Yufeng. Extensional concepts in intensional type theory, revisited. Theoretical Computer Science, 2025.

Two Notions of Equalities

Definitional	Propositional
$\vdash a_1 = a_2 : A$	$\vdash p : \mathsf{Id}_A(a_1, a_2)$

Dependent type theory with propositional equality gives intensional type theory (ITT).

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Equality reflection rule

Computation

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Adding equality reflection gives extensional type theory (ETT).

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Computation	Logic	Topology
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Substitution vs. transport

Definitional	Propositional
$t = t'$	$p : \mathsf{Id}(t, t')$
$B(t) = B(t')$	$B(t) \xrightarrow{p_*} B(t')$

Definitional vs. Propositional Equalities

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- Changing terms between types indexed by propositionally equal terms depends on the proof of equality.

Uniqueness of Identity Proofs (UIP)

$$\frac{\vdash p, p' : \mathsf{Id}_A(a_1, a_2)}{\vdash \mathsf{UIP}(p, p') : \mathsf{Id}(p, p')}$$	Uniqueness of identity proofs	Homotopically discrete space

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Theorem (Hofmann 1995)

ETT is conservative over ITT+UIP.

$$\frac{\vdash p, p' : \mathrm{Id}_A(a_1, a_2)}{\vdash \mathrm{UIP}(p, p') : \mathrm{Id}(p, p')} \quad \leftrightsquigarrow \quad \frac{\vdash p : \mathrm{Id}_A(a_1, a_2)}{\vdash a_1 = a_2 : A}$$

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Limitation. Syntactic result did not account for extensions.

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Need to Determine

1. What is a model of a type theory?
2. A suitable notion of equivalence between categories of models?

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Notation. If $\text{ft}\, A = \Gamma$ we write $A = \Gamma.A$.

Contextual Categories

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Substitutions

$$\begin{array}{ccc} \Delta.f^*A & \xrightarrow{f.A} & \Gamma.A \\ \pi\downarrow & \lrcorner & \downarrow\pi \\ \Delta & \xrightarrow[f]{} & \Gamma \end{array}$$

Homotopies in Contextual Categories

$$\frac{\vdash A \text{ Type}}{(x_1, x_2 : A) \vdash \mathrm{Id}_A(x_1, x_2) \text{ Type}}$$

Path object

Provable equality

Definition

A homotopy $H\colon f \sim g$ between $f, g\colon \Gamma \to \Delta \in \mathbb{C}$

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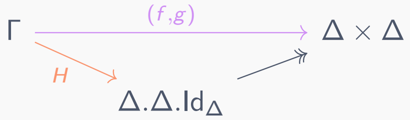

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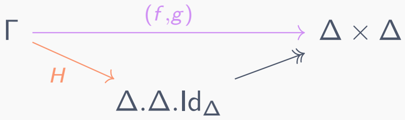

Homotopy equivalences $w\colon \Gamma \to \Delta$ are those maps admitting left and right homotopy inverses.

Theorem (Kapulkin–Lumsdaine 2018)

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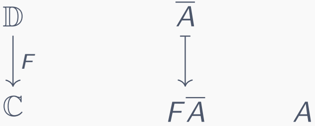

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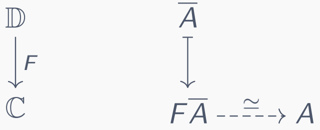

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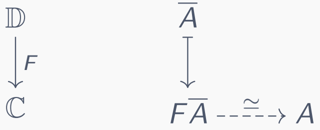

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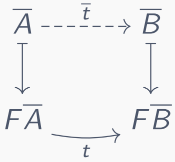

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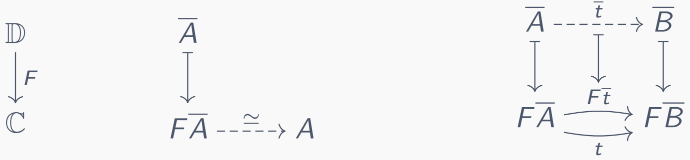

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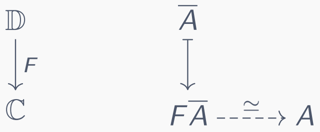

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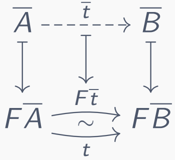

Conservativity of ITT+UIP over ETT, Revisited

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Two type theories $\mathbb{T}_1, \mathbb{T}_2$ extending ITT are Morita equivalent if there is a Quillen equivalence $\mathsf{CxlCat}_{\mathbb{T}_1} \underset{\longleftarrow}{\overset{\longrightarrow}{\bot}} \mathsf{CxlCat}_{\mathbb{T}_2}$.

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Example (Isaev 2020). The type theories ITT+Unit and ITT+Contr are Morita equivalent.

Theorem

The type theories ITT+UIP and ETT are Morita equivalent.

$$\mathsf{CxlCat}_{\mathsf{ITT+UIP}} \underset{|-|}{\overset{\langle - \rangle}{\rightleftarrows}} \mathsf{CxlCat}_{\mathsf{ETT}} \quad (\langle - \rangle \dashv |-|)$$

Proof.

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It suffices to check $\mathbb{C} \to |\langle \mathbb{C} \rangle|$ is a weak equivalence when $\mathbb{C} \in \mathsf{CxlCat}_{\mathsf{ITT+UIP}}$ is a cell-complex of the generating left class.

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Conservativity of ITT+UIP over ETT, Revisited

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From $\mathbb{C} \in \mathsf{CxlCat}_{\mathsf{ITT+UIP}}$ to $\langle\mathbb{C}\rangle \in \mathsf{CxlCat}_{\mathsf{ETT}}$

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Equality reflection ⇒ Identify homotopic terms

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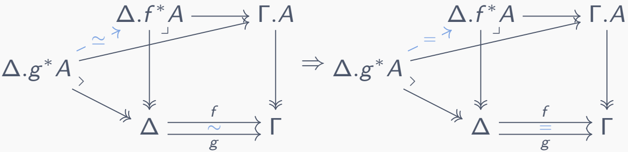

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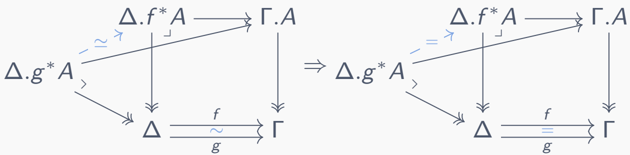
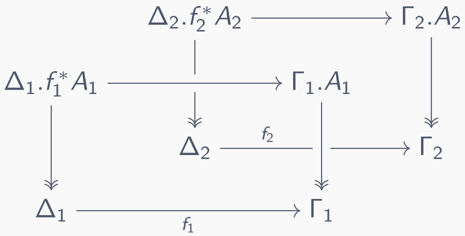


Identify types ⇒ Identify terms

From $\mathbb{C} \in \mathsf{CxlCat}_{\mathsf{ITT+UIP}}$ to $\langle \mathbb{C} \rangle \in \mathsf{CxlCat}_{\mathsf{ETT}}$

Repeatedly identify terms and types

Inductively collapse a wide subcategory of maps in $\mathcal{W}_{\mathsf{ETT}}$ to identities.



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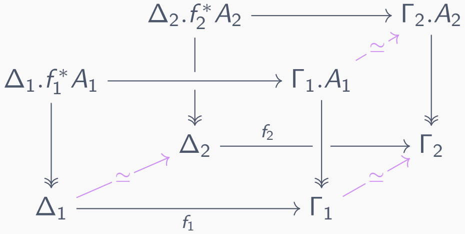

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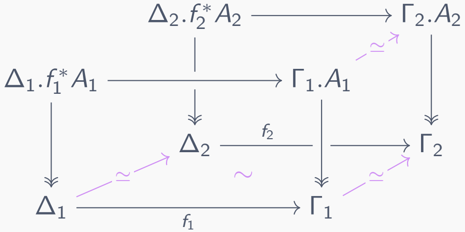

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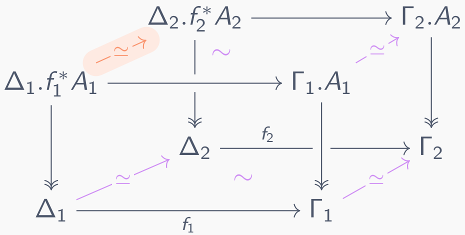

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- The solid maps above are in $\mathcal{W}_{\mathsf{ETT}}$; and
- The bottom face commute up to homotopy

From $\mathbb{C} \in \mathsf{CxlCat}_{\mathsf{ITT+UIP}}$ to $\langle\mathbb{C}\rangle \in \mathsf{CxlCat}_{\mathsf{ETT}}$

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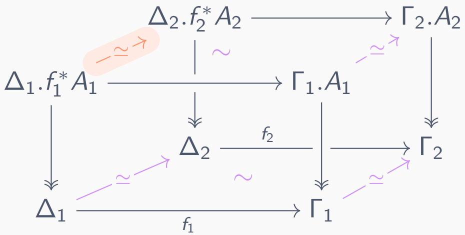

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This covers the freely-added base types $(\vec{x} : \Gamma) \vdash A(\vec{x})$.

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Example (Hofmann 1995). In the syntactic model of ITT + UIP, Hofmann calls $\mathsf{co} = \mathcal{W}_{\mathsf{ETT}}$.

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Example (Hofmann 1995). If $\mathbb{S}$ is the syntactic model, $\langle\mathbb{S}\rangle = \mathbb{Q}$ as from Hofmann.

Role of the UIP Axiom

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Define composition in $\langle \mathbb{C} \rangle$ and show well-definedness.

Proof. Replicate Hofmann's approach.

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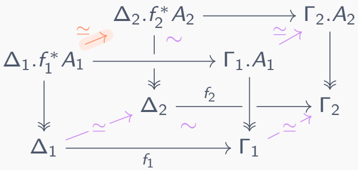

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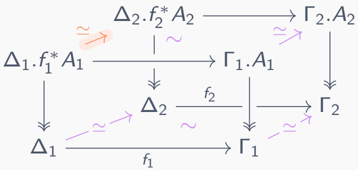

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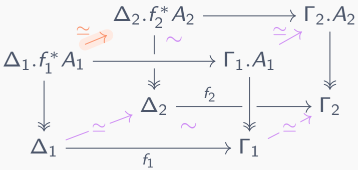

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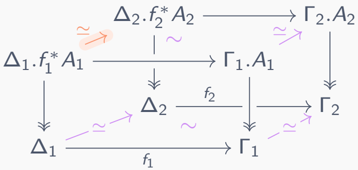

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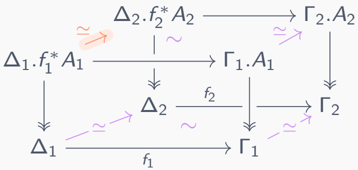

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Identification of Type Formers

Recall: identification of type formers

Suppose $w : \Gamma.A.B \simeq \Gamma'.A'.B' \in \mathcal{W}_{\mathsf{ETT}}$.

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We are done if prove $\mathcal{W}_{\mathsf{ETT}}$ cannot contain parallel maps between different type constructors (i.e. cases like $w_{\Sigma}(w) : \Gamma.\Pi(A, B) \simeq \Gamma'.\Pi(A', B') \in \mathcal{W}_{\mathsf{ETT}}$).

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Weak Head Normal Form in Cellular Models

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Suppose $\mathbb{C}$ is cellular.

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Proposition

Suppose $\mathbb{C}$ is cellular. Then, there is a set of base types $\{(\Theta_i \in \mathbb{C}, T_i \in \mathsf{Ty}_{\mathbb{C}}\Theta_i)\}_{i \in I}$

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Suppose $\mathbb{C}$ is cellular. Then, there is a set of base types $\{(\Theta_i \in \mathbb{C}, T_i \in \mathsf{Ty}_{\mathbb{C}}\Theta_i)\}_{i \in I}$ such that if $X \in \mathsf{Ty}_{\mathbb{C}}\Gamma$ then precisely one of the following cases are true:

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Proof. By cellular filtration. ♡

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By UIP, if $w, w' : X \simeq X' \in \mathcal{W}_{\mathsf{ETT}}$ then $w \simeq w'$.

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If $\mathbb{C} \in \mathsf{CxlCat}_{\mathsf{ITT+UIP}}$ then the quotient category $\langle \mathbb{C} \rangle \in \mathsf{CxlCat}_{\mathsf{ETT}}$ is a category with well-defined composition.

Summary

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Theorem

The type theories ITT+UIP and ETT are Morita equivalent.

$$\mathsf{CxlCat}_{\mathsf{ITT+UIP}} \underset{|-|}{\overset{\langle - \rangle}{\underset{\perp}{\rightleftarrows}}} \mathsf{CxlCat}_{\mathsf{ETT}}$$

Future directions

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Thank you!