Towards a type theory for (∞, ω)-categories

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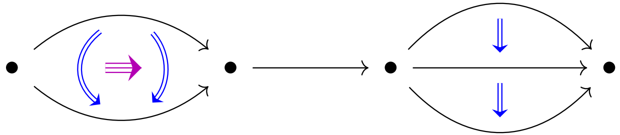
Main Idea

- Simplicial Type Theory (*STT*)
  (E. Riehl, M. Shulman, D. Gratzer, J. Weinberger, U. Buchholtz,)
  - $[\![\mathcal{U}]\!] = \mathbf{Psh}(\mathbf{\Delta})$
  - $(\infty, 1)$-categories = Complete Segal Spaces.

- Cellular Type Theory (*CellTT*)
  (Using C. Rezk Θ-Spaces + F. Loubaton Thesis)
  - $[\![\mathcal{U}]\!] = \mathbf{Psh}(\mathbf{\Theta})$
  - (∞, ω)-categories = $\mathbf{\Theta}-$Spaces.

The category Θ

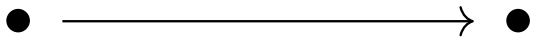
- Objects: Pasting schemes.



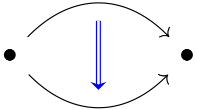
- Morphisms: Morphisms of strict ω-categories.

Some objects of Θ

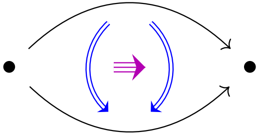
$\mathcal{O}_0 = *$



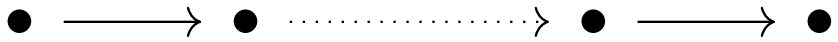
$\mathcal{O}_1 = \Delta^1$



$\mathcal{O}_2$



$\mathcal{O}_3$

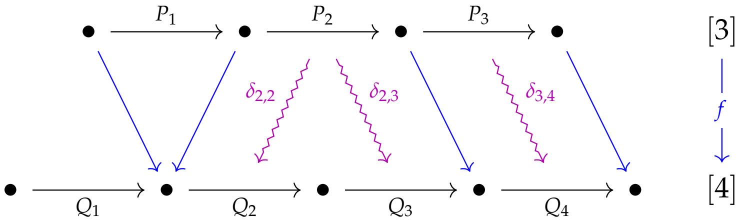


Δ^n

A combinatorial description of Θ

- Objects are lists of objects.

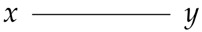
- Morphisms:



Hom Types in STT

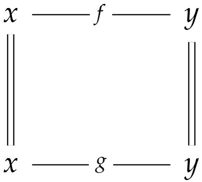
In STT:

$$\mathsf{hom}_A(x, y) = \sum_{f : \mathsf{I} \to A} (f\,0 = x) \times (f\,1 = y)$$



Then:

$$\mathsf{hom}_{\mathsf{hom}_A(x,y)}(f, g) = \sum_{H : \mathsf{I}^2 \to A} \cdots$$



Does not generalize to Θ

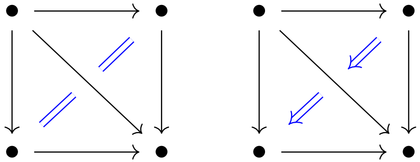
$$\mathsf{I} = よ(\mathcal{O}_1)$$

2-cells of I^2 are pairs (x, y) of 2-cells of I.

x, y are invertible $\Rightarrow (x, y)$ too.

Hence I^2 is 1-categorical

$$\mathsf{hom}_A(x, y) \neq \sum_{f : \mathsf{I} \to A} (f\,0 = x) \times (f\,1 = y)$$



Another approach

Workaround:

$$\mathbf{Psh}(\mathbf{\Theta}) \underset{\mathsf{hom}}{\overset{\$}{\underset{\bot}{\rightleftarrows}}} \mathbf{Psh}(\mathbf{\Theta})_{\bullet\bullet}$$

Two subgoals:

- Defining a suspension \$.
- Postulating the adjunction.

($ ⊣ hom) adjunction

$$\underbrace{(A \to \mathsf{hom}_B(x, y))}_{\text{internal hom}} \neq \underbrace{(\$A \to_{\bullet\bullet} (B, x, y))}_{\text{internal hom}}$$

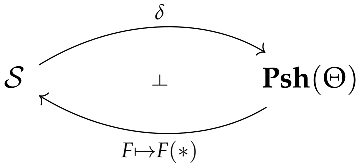
$$\flat(A \to \mathsf{hom}_B(x, y)) = \flat(\$A \to_{\bullet\bullet} (B, x, y))$$

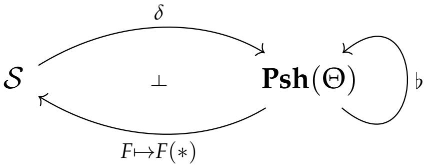
The $\flat$ modality

$$A :: \mathcal{U} \quad \rightsquigarrow \quad [\![A]\!] \in \mathbf{Psh}(\boldsymbol{\Theta})$$

$$\flat A : \mathcal{U} \quad \rightsquigarrow \quad [\![\flat A]\!] = [\![A]\!](*)$$

Semantic of $\flat$





Crisp Type Theory

Why Crisp Type Theory ?

Because $\flat$ is not “continuous”.

Two kinds of hypothesis:

continuous	$X : \mathcal{U}, x : X$
crisp	$X :: \mathcal{U}, x :: X$

$$\frac{\Gamma | \cdot \vdash X : \mathcal{U}}{\Gamma | \cdot \vdash \flat X : \mathcal{U}}$$

CellTT

$$\text{CellTT} = \underbrace{\text{HoTT} \ + \ \text{Idempotent comodality}}_{\text{Crisp Type Theory}} + \text{Axioms}$$

Pasting Schemes

$$\text{pasting schemes:} \quad \begin{cases} \mathsf{PS} : \mathsf{Set} \\ [] : \mathsf{Array}\,\mathsf{PS} \to \mathsf{PS} \end{cases}$$

$$\text{morphisms:} \quad P \to_{\mathsf{PS}} Q : \mathsf{Set} \qquad (P, Q : PS)$$

$$\text{suspension:} \quad \begin{cases} \$: \mathsf{PS} & \to & \mathsf{PS} \\ P & \to & [P] \end{cases}$$

$$* = [] \quad \Delta^n = [*, \cdots, *]$$

$$\mathcal{O}_n = \$^n[] = [[\cdots[]\cdots]]$$

Yoneda Embedding

$$\text{Yoneda:} \quad よ : \mathsf{PS} \to \mathcal{U}$$

$$X :: \mathcal{U} \quad \rightsquigarrow \quad X_P :\equiv \flat(よ(P) \to X)$$

$$f :: X \to Y \quad \rightsquigarrow \quad f_P : X_P \to Y_P$$

$$\sigma : P \to_{\mathsf{PS}} Q \quad \rightsquigarrow \quad \sigma^* : X_Q \to X_P$$

Some Axioms

- Equivalences are pointwise

$$\left(\prod_{P:PS} \mathsf{is\text{-}equiv}(f_P)\right) \to \mathsf{is\text{-}equiv}(f)$$

- よ is fully faithfull
- $(-)_P$ preserves colimits
- $\left(\sum_{P:PS} \sum_{c:X_P} よP\right) \to X$ is an effective epi ((-1)-truncated)
- $\flat$-discreteness is *cellular discreteness*:

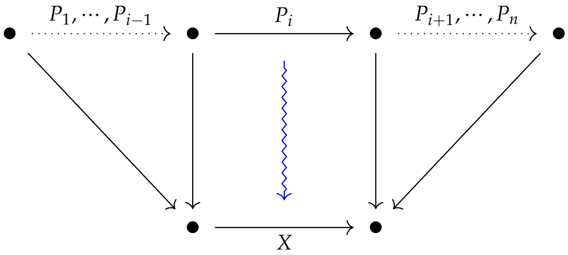
$$\mathsf{is\text{-}equiv}(\flat X \to X) \leftrightarrow \prod_{P:PS} \mathsf{is\text{-}equiv}(X \to (よ(P) \to X))$$

Suspension

We extend \$ to $\mathcal{U}$.

- $よ(\$P) = \$(よP)$
- If $P = [P_1, P_2, \cdots, P_n]$:

$$(\$X)_P \cong \mathbb{1} + \left(\sum_{i : \mathsf{Fin}_n} X_{P_i} \right) + \mathbb{1}$$



Hom Types

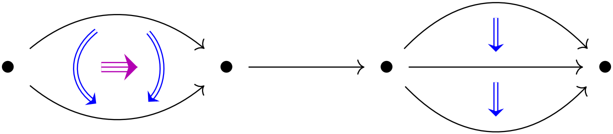
We postulate an adjunction

$$\flat(A \to \mathsf{hom}_B(x, y)) \cong \flat(\$A \to_{\bullet\bullet} (B, x, y))$$

Cellular Realization

$$|[P_1, \cdots, P_m]| :\equiv$$

$$\text{colim} \quad \begin{array}{ccccccc} & \mathbb{1} & & \mathbb{1} & & \mathbb{1} & \\ \swarrow & & \searrow \swarrow & & \searrow \swarrow & & \searrow \\ \$|P_1| & & \$|P_2| & & \cdots & & \$|P_m| \end{array}$$



Segal Types

We define a map $|P| \to よP$

We define is-Segal:

$$\flat(よP \to X) \xrightarrow{\sim} \flat(|P| \to X)$$

(∞, ω)-categories

As in Riehl-Shulman STT, there is a completeness condition.
$\mathsf{is\text{-}complete}(X)$

$$(\infty, \omega)\text{-categories} = \text{complete Segal Types.}$$

$$(\infty, \omega)\text{-}\mathsf{Cat} = \sum_{X:\mathcal{U}} \mathsf{is\text{-}Segal}(X) \times \mathsf{is\text{-}complete}(X)$$

What's next ?

Main goal: Proving a Yoneda Lemma.

- Defining a well-suited notion of fibration
- Working out properties of Segalness and completeness

Currently:

$$\mathsf{is\text{-}Segal}(X) \to \prod_{x,y:X} \mathsf{is\text{-}Segal}\left(\mathsf{hom}_X(x, y)\right)$$

The end

Thank you !

Questions ?