Towards computing $\pi_6\mathbb{S}^4$ in HoTT

Beyond EHP via the relative James construction

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Our result

Theorem

For some $n \in \{1, 2\}$, *we have an isomorphism* $\pi_5 \mathbb{S}^3 \cong \mathbb{Z}/n\mathbb{Z}$.

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Motivation

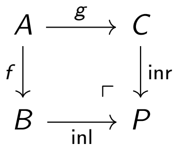
By Freudenthal, $\pi_6\mathbb{S}^4$ is equivalent to the second stable homotopy group of spheres: $\pi_2^S :\equiv \lim_{n\to\infty} \pi_{n+2}\mathbb{S}^n$.

Outline

1. Background
   - Basic definitions
   - Some history of $\pi_k \mathbb{S}^n$
   - Whitehead's EHP exact sequence
   - Gray's relative James construction

2. Our result
   - Main lemma: Gray's corollary
   - Application to $\pi_5 \mathbb{S}^3 \cong \pi_6 \mathbb{S}^4 \cong \mathbb{Z}/n\mathbb{Z}$
   - Future work?

Pushouts



Definition

Given maps $f : A \to B$, $g : A \to C$, we define the pushout P as a higher inductive type:

$$\mathsf{inl} : B \to P \quad \mathsf{inr} : C \to P$$

$$\mathsf{push} : \Pi_{a:A}\, \mathsf{inl}\,(f\,(a)) = \mathsf{inr}\,(g\,(a))$$

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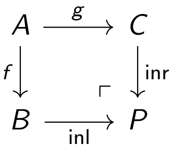

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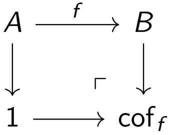

Definition

Given a map $f : A \to B$, we define the cofiber cof_f as the pushout of the map $A \to 1$ with f.

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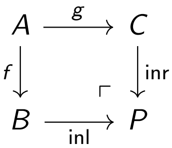

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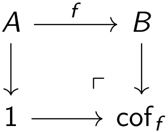

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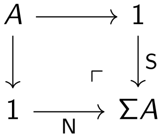

Definition

Given a type A, we define the suspension ΣA as the pushout of the map $A \to 1$ with itself. We write $\mathsf{merid} : A \to \mathsf{N} =_{\Sigma A} \mathsf{S}$.

Spheres and homotopy groups

Definition

We define the *n-sphere* $\mathbb{S}^n$ as an iterated suspension:

$$\mathbb{S}^0 :\equiv \text{bool}$$
$$\mathbb{S}^{n+1} :\equiv \Sigma\mathbb{S}^n$$

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Given a pointed type A, we define the *nth loop space $\Omega^n A$* as the type of pointed maps $\mathbb{S}^n \to_\bullet A$, itself pointed by the constant map at a_0.

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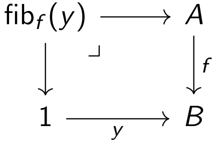
Definition

Given a pointed type A, we define its n-th homotopy group $\pi_n A$ $(n \geq 1)$ as the set truncation $\|\Omega^n A\|_0$. We give this a group structure equivalent to composition on iterated identity types...

Fibers and connectedness

Definition

Given a map $f : A \to B$ and a point $y : B$, we define the fiber $\mathrm{fib}_f(y) :\equiv \Sigma_{x:A}\, f(x) = y$. When B is pointed we may just write fib_f.



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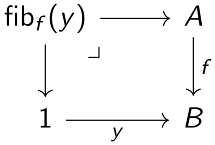

Definition

We say a type A is *n*-connected if its *n*-truncation is contractible: $\mathrm{isConnected}_n(A) :\equiv \mathrm{isContr}\left(\|A\|_n\right)$.

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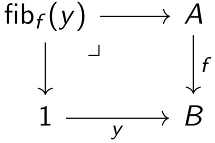

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Definition

We say a map $A \to B$ is *n*-connected if all its fibers are *n*-connected: $\Pi_{y:B}\ \mathrm{isConnected}_n(\mathrm{fib}_f(y))$.

The long exact sequence in homotopy groups

Lemma (UF13, 8.4.6)

If f is n-connected, then it induces isomorphisms $\pi_k A \cong \pi_k B$ for $k \leq n$.

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Theorem (UF13, 8.4.6)

Given a pointed map $f : A \to_\bullet B$, we have an exact sequence in homotopy groups:

$$\begin{array}{lllllll} & \dots & & & & \\ \to & \pi_k \mathrm{fib}_f & \to & \pi_k A & \to & \pi_k B \\ \to & \dots & & & & \\ \to & \pi_1 \mathrm{fib}_f & \to & \pi_1 A & \to & \pi_1 B \end{array}$$

$\pi_k \mathbb{S}^n$ in HoTT/UF

Theorem (Licata and Shulman 2013, Licata and Brunerie 2013)

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For some $n \in \mathbb{Z}, \pi_4 \mathbb{S}^3 \cong \mathbb{Z}/n\mathbb{Z}$. *Also,* $n = 2$.

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Corollary

- $\pi_1^S :\equiv \lim_{n\to\infty} \pi_{n+1}\mathbb{S}^n \cong \mathbb{Z}/2\mathbb{Z}$
- $\pi_5\mathbb{S}^4 \cong \mathbb{Z}/2\mathbb{Z}$
- $\pi_4\mathbb{S}^2 \cong \mathbb{Z}/2\mathbb{Z}$

$\pi_4\mathbb{S}^3$ in `Cubical Agda`

Theorem (Ljungström and Mörtberg 2023, mechanised and computer-assisted in `Cubical Agda`)

For some $n \in \mathbb{Z}$, $\pi_4\mathbb{S}^3 \cong \mathbb{Z}/n\mathbb{Z}$, and the computer says refl : $n \equiv 2$.

Whitehead's EHP exact sequence

Theorem (Whitehead 1953, but see Devalapurkar and Haine 2021)

We have an exact sequence:

$$\begin{array}{cccccc} & \pi_{3n-2}\mathbb{S}^n & \to & \ldots & & \\ \to & \pi_q\mathbb{S}^n & \xrightarrow{\mathsf{E}} & \pi_{q+1}\mathbb{S}^{n+1} & \xrightarrow{\mathsf{H}} & \pi_q\mathbb{S}^{2n} \\ \xrightarrow{P} & \pi_{q-1}\mathbb{S}^n & \to & \ldots & & \end{array}$$

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Corollary (See Hatcher 2002 or Whitehead 1978)

We have a surjection $\pi_4\mathbb{S}^2 \twoheadrightarrow \pi_5\mathbb{S}^3$*, from:*

$$\begin{array}{cccccc} & \pi_4\mathbb{S}^2 & \xrightarrow{\mathsf{E}} & \pi_5\mathbb{S}^3 & \xrightarrow{\mathsf{H}} & \pi_4\mathbb{S}^4 \\ \xrightarrow{P} & \pi_3\mathbb{S}^2 & \to & \pi_4\mathbb{S}^3 & \to & 1 \end{array}$$

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Remark

Cagne et al. 2024 claim their exact sequence can reproduce Whitehead's EHP, following Lang 1973. It should also extend beyond EHP somehow...

Gray's relative James construction: the pinch map

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \longrightarrow & 1 \\ \downarrow & & \downarrow & & \downarrow \\ 1 & \longrightarrow & \mathrm{cof}_f & \xrightarrow[\mathrm{pinch}_f]{} & \Sigma A \end{array}$$

Definition

Given a map $f : A \to B$, we define the "pinch map" $\mathrm{pinch}_f : \mathrm{cof}_f \to \Sigma A$ as follows:

$$\mathrm{pinch}_f(\mathrm{inl}(*)) :\equiv \mathsf{N} \quad \mathrm{pinch}_f(\mathrm{inr}(b)) :\equiv \mathsf{S}$$
$$\mathrm{ap}_{\mathrm{pinch}_f}(\mathrm{push}(a)) :\equiv \mathrm{merid}(a)$$

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Remark

We now want to understand the fiber of the pinch map $\mathrm{fib}_{\mathrm{pinch}_f}$, in order to get information from the long exact sequence in homotopy groups:

$$\ldots \to \pi_k \mathrm{fib}_{\mathrm{pinch}_f} \to \pi_k \mathrm{cof}_f \overset{\pi_k \mathrm{pinch}_f}{\to} \pi_k \Sigma A \to \ldots$$

Classically, Gray 1973 introduced this technique. Related but more sophisticated techniques have been used recently by homotopy theorists: see Yang, Mukai, and Wu 2024, Zhu and Jin 2024, Zhu 2024.

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There is a relative James construction, which is equivalent to the fiber of the pinch map.

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When the types are suspensions, the second stage of the relative James filtration is equivalent to the cofiber of a certain generalized Whitehead product.

Remark

Baker's "fiber HIT heuristic" (touched on in Baker 2024) should produce a relative James construction HIT.

Main lemma: statement

Lemma (Adapted from Gray 1973, corollary 5.8)

Suppose given pointed types A, B and a map $f : \Sigma A \to \Sigma B$. If A is $(a-1)$-connected, then we have a $2a$-connected map

$$\gamma : \mathsf{cof}_{[\mathsf{id}_{\Sigma B}, f]} \to \mathsf{fib}_{\mathsf{pinch}_f}$$

Main lemma: proof sketch

We construct two pushout squares:

$$
\begin{array}{ccc}
\Sigma A \vee \Sigma A & \xrightarrow{\nabla} & \Sigma A \\
{\scriptstyle (f\times \mathrm{id})\circ\iota^{\vee}}\downarrow & \ulcorner & \downarrow \\
\Sigma B \times \Sigma A & \longrightarrow & \mathrm{cof}_{[\mathrm{id}_{\Sigma B}, f]}
\end{array}
\qquad
\begin{array}{ccc}
\Sigma A \times \Omega\Sigma^2 A & \xrightarrow{(x,p)\mapsto\sigma(x)\cdot p} & \Omega\Sigma^2 A \\
{\scriptstyle f\times \mathrm{id}}\downarrow & \ulcorner & \downarrow \\
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We then define a map $\gamma : \mathrm{cof}_{[\mathrm{id}_{\Sigma B}, f]} \to \mathrm{fib}_{\mathrm{pinch}_f}$ induced by a map of spans:

$$\begin{array}{ccccc} \Sigma B \times \Sigma A & \longleftarrow & \Sigma A \vee \Sigma A & \longrightarrow & \Sigma A \\ \downarrow{\scriptstyle \mathrm{id}\times\sigma_{\Sigma A}} & & \downarrow{\scriptstyle (\mathrm{id}\times\sigma_{\Sigma A})\circ\iota^{\vee}} & & \downarrow{\scriptstyle \sigma_{\Sigma A}} \\ \Sigma B \times \Omega\Sigma^2 A & \longleftarrow & \Sigma A \times \Omega\Sigma^2 A & \longrightarrow & \Omega\Sigma^2 A \end{array}$$

Since $\sigma_{\Sigma A} : \Sigma A \to \Omega\Sigma^2 A$ and $\iota^{\vee} : \Sigma A \vee \Sigma A \to \Sigma A \times \Sigma A$ are $2a$-connected, all three vertical maps will be $2a$-connected, and so γ will be too.

Applying the main lemma

Recall the Brunerie element $[\iota_2, \iota_2] : \mathbb{S}^3 \to_\bullet \mathbb{S}^2$ (writing $\iota_2 :\equiv \mathsf{id}_{\mathbb{S}^2}$.) We will apply the main lemma to this map.

Lemma (Brunerie 2016)

We have a 4-connected map $\mathsf{cof}_{[\iota_2,\iota_2]} \to \Omega\mathbb{S}^3$. *In particular:*

$$\pi_3 \mathsf{cof}_{[\iota_2,\iota_2]} \cong \pi_4 \mathbb{S}^3$$
$$\pi_4 \mathsf{cof}_{[\iota_2,\iota_2]} \cong \pi_5 \mathbb{S}^3$$

Applying the main lemma

Construction

Applying the main lemma with $\mathsf{pinch}_{[\iota_2,\iota_2]} : \mathsf{cof}_{[\iota_2,\iota_2]} \to \mathbb{S}^4$, we get a 4-connected map $\gamma : \mathsf{cof}_{[\iota_2,[\iota_2,\iota_2]]} \to \mathsf{fib}_{\mathsf{pinch}_{[\iota_2,\iota_2]}}$. This gives an exact sequence:

$$
\begin{array}{ccccccc}
 & & & & \pi_5\mathsf{cof}_{[\iota_2,\iota_2]} & \to & \pi_5\mathbb{S}^4 \\
\to & \pi_4\mathsf{cof}_{[\iota_2,[\iota_2,\iota_2]]} & \to & \pi_4\mathsf{cof}_{[\iota_2,\iota_2]} & \to & \pi_4\mathbb{S}^4 \\
\to & \pi_3\mathsf{cof}_{[\iota_2,[\iota_2,\iota_2]]} & \to & \pi_3\mathsf{cof}_{[\iota_2,\iota_2]} & \to & 1
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Lemma

Now, using results by Brunerie and (very recently!) by Ljungström:

$$[\iota_2,[\iota_2,\iota_2]] = [\iota_2, \pm 2\eta] = \pm 2[\iota_2,\eta]$$

which must be trivial since $\pi_4 \mathbb{S}^2 \cong \mathbb{Z}/2\mathbb{Z}$. Therefore $\mathsf{cof}_{[\iota_2,[\iota_2,\iota_2]]} \simeq \mathbb{S}^2 \vee \mathbb{S}^5$ (at least merely...)

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which must be trivial since $\pi_4 \mathbb{S}^2 \cong \mathbb{Z}/2\mathbb{Z}$. *Therefore* $\mathsf{cof}_{[\iota_2,[\iota_2,\iota_2]]} \simeq \mathbb{S}^2 \vee \mathbb{S}^5$ *(at least merely...)*

Applying the main lemma

Proof of main theorem.

We now have this EHP-like exact sequence:

$$
\begin{array}{cccccc}
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\begin{array}{ccccccc}
 & & & & & & \pi_5\mathbb{S}^4 \\
\xrightarrow{\partial} & \pi_4\mathbb{S}^2 & \twoheadrightarrow & \pi_5\mathbb{S}^3 & \xrightarrow{0} & \pi_4\mathbb{S}^4 \\
\xhookrightarrow{\times 2} & \pi_3\mathbb{S}^2 & \overset{\%2}{\twoheadrightarrow} & \pi_4\mathbb{S}^3 & \rightarrow & 1
\end{array}
$$

Applying the main lemma

Proof of main theorem.

We now have this EHP-like exact sequence:

$$\begin{array}{ccccccc} & & & & & & \pi_5\mathbb{S}^4 \\ \xrightarrow{\partial} & \pi_4\mathbb{S}^2 & \twoheadrightarrow & \pi_5\mathbb{S}^3 & \xrightarrow{0} & \pi_4\mathbb{S}^4 \\ \xhookrightarrow{\times 2} & \pi_3\mathbb{S}^2 & \overset{\%2}{\twoheadrightarrow} & \pi_4\mathbb{S}^3 & \rightarrow & 1 \end{array}$$

Since both $\pi_5\mathbb{S}^4$ and $\pi_4\mathbb{S}^2$ are isomorphic to $\mathbb{Z}/2\mathbb{Z}$, we are done: we have some $\partial : \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$, and we define $n :\equiv \partial(1) \in \{1, 2\}$. Then since $\pi_5\mathbb{S}^3 \cong (\mathbb{Z}/2\mathbb{Z})/\mathrm{Im}(\partial)$, we have $\pi_5\mathbb{S}^3 \cong \mathbb{Z}/n\mathbb{Z}$. □

Future work?

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- Mechanise this in `Cubical Agda`!
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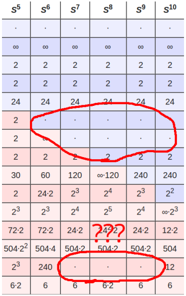
- What's going on here?

$$0 \cong \pi_4^S \cong \pi_5^S \cong \pi_{12}^S$$
$$\cong \pi_{19}\mathbb{S}^7 \cong \pi_{20}\mathbb{S}^8 \cong \pi_{21}\mathbb{S}^9$$



S^5	S^6	S^7	S^8	S^9	S^{10}
·	·	·	·	·	·
∞	∞	∞	∞	∞	∞
2	2	2	2	2	2
2	2	2	2	2	2
24	24	24	24	24	24
2	·	·	·	·	·
2	∞	·	·	·	·
2	2	2	2	2	2
30	60	120	∞·120	240	240
2	24·2	2^3	2^4	2^3	2^2
2^3	2^3	2^4	2^5	2^4	∞·2^3
72·2	72·2	24·2	24·2	24·2	12·2
504·2^2	504·4	504·2	504·2	504·2	504
2^3	240	·	·	·	12
6·2	6	6	6·2	6	6

Figure: Wikipedia

Questions?

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