About the construction of simplicial and cubical sets in indexed form: the case of degeneracies

Hugo Herbelin
joint work with Ramkumar Ramachandra

HoTT/UF 2025

Genova

15 April 2025

Outline

- Reedy presheaves in (usual) fibered form vs indexed form, why?

- unary and binary parametricity as a language to uniformly talk about respectively augmented simplicial and cubical sets

- simplicial degeneracies are actually not a unary form of cubical degeneracies but of connections

- an effective uniform indexed construction of augmented simplicial and cubical sets with one degeneracy (machine-checked in Rocq)

The fibred/indexed correspondence for h-sets

For B : HSet_l

$$\Sigma E : \mathsf{HSet}_l.\,(E \to B) \;\simeq\; B \to \mathsf{HSet}_l$$

Application to the definition of Reedy presheaves in *indexed form*, here for cubical sets:

fibred form	vs	*indexed form*	
$Y_0 \; : \; \mathsf{HSet}_l$		$X_0 \; : \; \mathsf{HSet}_l$	(points)
$\uparrow\uparrow$			
$Y_1 \; : \; \mathsf{HSet}_l$		$X_1 \; : \; X_0 \times X_0 \to \mathsf{HSet}_l$	(segments)
$\uparrow\uparrow\uparrow\uparrow$			
$Y_2 \; : \; \mathsf{HSet}_l$		$X_2 \; : \; \Pi(x_{LL}, x_{LR}) : (X_0 \times X_0).\, \Pi x_{L*} : X_1(x_{LL}, x_{LR}).$	
		$\Pi(x_{RL}, x_{RR}) : (X_0 \times X_0).\, \Pi x_{R*} : X_1(x_{RL}, x_{RR}).$	
+ coherences		$X_1(x_{LL}, x_{RL}) \times X_1(x_{LR}, x_{RR}) \to \mathsf{HSet}_l$	(squares)
$\vdots$		$\vdots$	

Iterating the fibred/indexed correspondence

The domain of such X_n is a *matching object* of Y at n, say M_n, so we can build such X from such Y over a direct category by making a choice M of "matching objects". Let's write:

- c_n for the map from Y_n to M_n
- π_f for the projection to Y_p associated to the face map $f : p \to n$ $(p \neq n)$
- M_f for the projection to M_p associated to the face map $f : p \to n$

Then, we have the following maps back and forth (producing back a *Reedy fibrant* Y):

fibred form		*indexed form*
Y	$\mapsto$	$X_n \triangleq \lambda d : M_n . \Sigma y : Y_n . (c_n(y) = d)$
$\left(\begin{array}{lcl} Y_n & \triangleq & \Sigma d : M_n .\, X_n(d) \\ Y_{id} & \triangleq & id \\ Y_{f:p\to n} & \triangleq & \lambda(d, \_).\, (M_f(d), \pi_f(d)) \end{array}\right)$	$\leftarrow\!\shortmid$	X

A precise study to be done, showing that we get an equivalence (not the purpose of the talk though).

Building presheaves in indexed form directly

Alternatively, we can define "matching objects" $M_n(X_0, ..., X_{n-1})$ directly on the indexed side without referring first to the fibred side. This was done, e.g., for semi-simplicial sets:

- By defining matching object as the collection of all faces, quotiented with $M_{f \circ g} = M_f \circ M_g$, as in Voevodsky 2012, Part and Luo 2015, Altenkirch, Capriotti and Kraus 2016, ...
- By relying on specific presentations of a category:
  - The d_i^n generators and $d_i d_j = d_{j-1} d_i$ coherences in H. 2013
  - By following parametricity rules in H. and Ramachandra 2025

Eventually expecting interpretations, e.g. of the universe, that more closely follow the syntax...

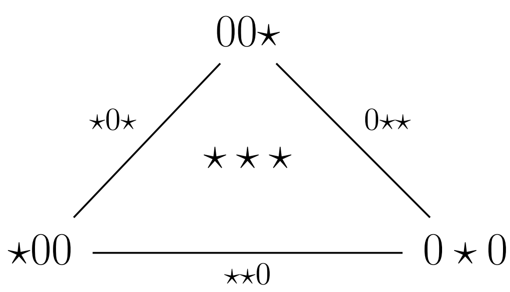
Iterated parametricity as a uniform approach to both augmented simplicial sets and cubical sets

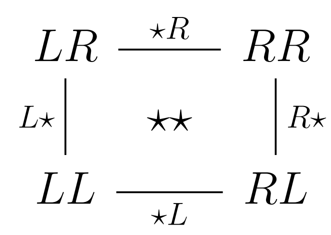
It seems now established that the augmented simplicial and cubical categories only differ in the "arity" of a finite set ν:

$$
\begin{array}{lll}
\mathsf{Obj} & := & \mathbb{N} \\
\mathsf{Hom}(p,n) & := & \{l \in (\nu \sqcup \{\star\})^n \mid \text{number of } \star \text{ in } l = p\} \\
g \circ f & := & \begin{cases} f & \text{if } g = \epsilon \\ a\,(g' \circ f) & \text{if } g = a\,g', \text{where } a \in \nu \\ a\,(g' \circ f') & \text{if } g = \star\,g',\ f = a\,f', \text{ where } a \in \nu \text{ or } a = \star \end{cases} \\
\mathsf{id} & := & \star \ldots \star \;\; n \text{ times for } \mathsf{id} \in \mathsf{Hom}(n,n)
\end{array}
$$

That is, we obtain:

augmented semi-simplicial sets with $\nu = \{0\}$ | semi-cubical sets with $\nu = \{L, R\}$

e.g.: 

e.g.: 

Adding (one) reflexivity (in the last direction)

fibred form	*vs*	*indexed form*	
$Y_0 \;:\; \mathsf{HSet}_l$		$X_0 \;:\; \mathsf{HSet}_l$	(points)
$\uparrow\uparrow\downarrow$			
$Y_1 \;:\; \mathsf{HSet}_l$		$X_1 \;:\; X_0 \times X_0 \to \mathsf{HSet}_l$	(segments)
$\uparrow\uparrow\uparrow\uparrow\downarrow$		$r_1 \;:\; \Pi x_0 : X_0.\, X_1(x_0, x_0)$	
$Y_2 \;:\; \mathsf{HSet}_l$		$X_2 \;:\; \Pi(x^0_{LL}, x^0_{LR}) : (X_0 \times X_0).\, \Pi x^1_{L*} : X_1(x^0_{LL}, x^0_{LR}).$ $\Pi(x^0_{RL}, x^0_{RR}) : (X_0 \times X_0).\, \Pi x^1_{R*} : X_1(x^0_{RL}, x^0_{RR}).$	
+ coherences		$X_1(x^0_{LL}, x^0_{RL}) \times X_1(x^0_{LR}, x^0_{RR}) \to \mathsf{HSet}_l$	(squares)
		$r_2 \;:\; \Pi(x^0_L, x^0_R) : (X_0 \times X_0).\, \Pi x^1 : X_1(x^0_L, x^0_R).$ $X_2((x^0_L, x^0_L), r_1(x^0_L), (x^0_R, x^0_R), r_1(x^0_R), (x^1, x^1))$	
$\vdots$		$\vdots$	

(other reflexivities can be obtained if we add also, e.g., permutations)

In terms of matching objects, reflexivities have the form:

$$r_n : \Pi d : M_n.\, \Pi x : X_n(d).\, X_{n+1}(M_r(d, x))$$

for some $M_r : (\Sigma d : M_n.\, X_n(d)) \to M_{n+1}$ to be defined

In passing: simplicial degeneracies are *not* the unary case of cubical degeneracies but the unary case of cubical *connections*

	set	*reflexivities* (*involve one direction*)	*connections* (*involve two directions*)
unary	X_{-1} : HSet $X_0 : X_{-1} \to$ HSet $X_1 : \Pi x^{-1}.\, X_0(x^{-1}) \to X_0(x^{-1}) \to$ HSet	$r_{-1} : \Pi x^{-1}.X_0(x^{-1})$ $r_0 : \Pi x^{-1}.\, \Pi x^0.\, X_1(r_{-1}(x^{-1}), x^0)$ (as in Parametric Type Theory)	$c_0 : \Pi x^{-1}.\, \Pi x^0.\, X_1(x^0, x^0)$
binary	X_0 : HSet$_l$ $X_1 : X_0 \times X_0 \to$ HSet$_l$ $X_2 : \Pi(x^0_{LL}, x^0_{LR})x^1_{L\star}(x^0_{RL}, x^0_{RR})x^1_{R\star}.\, X_1(x^0_{LL}, x^0_{RL}) \times X_1(x^0_{LR}, x^0_{RR}) \to$ HSet$_l$	$r_0 : \Pi x^0.X_1(x^0, x^0)$ $r_1 : \Pi(x^0_L, x^0_R).\, \Pi x^1.\, X_2((x^0_L, x^0_L), r_0(x^0_L), (x^0_R, x^0_R), r_0(x^0_R), (x^1, x^1))$	$c_{1L} : \Pi(x^0_L, x^0_R).\, \Pi x^1.\, X_2((x^0_L, x^0_L), r_0(x^0_L), (x^0_L, x^0_R), x^1, (r_0(x^0_L), x^1))$
n-ary		(only one reflexivity per direction)	(n connections per direction, completing full arity with $n-1$ reflexivities)

An effective indexed construction as a dependent stream of dependent sets

ν-sets

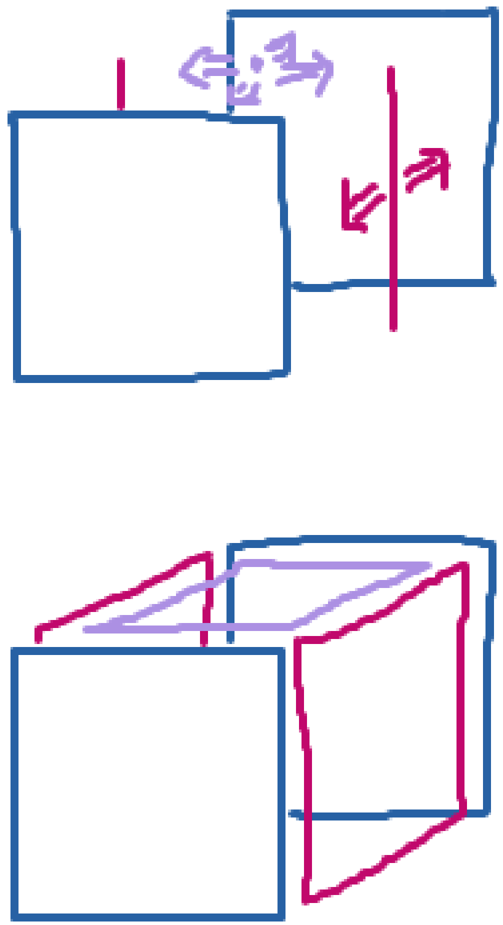
$$
\begin{array}{lll}
\nu\mathsf{set}_l & & : \ \mathsf{Type}_{l+1} \\
\nu\mathsf{set}_l & & \triangleq \ \nu\mathsf{set}_l^{\geq 0}(\star) \\
\\
\nu\mathsf{set}_l^{\geq n} & (X_{<n} : \nu\mathsf{set}_l^{<n}) & : \ \mathsf{Type}_{l+1} \\
\nu\mathsf{set}_l^{\geq n} & X_{<n} & \triangleq \ \Sigma X_n : \nu\mathsf{set}_l^{=n}(X_{<n}).\, \nu\mathsf{set}_l^{\geq n+1}(X_{<n}, X_n)
\end{array}
$$

Truncated ν-sets

$$
\begin{array}{lll}
\nu\mathsf{set}_l^{<n} & & : \ \mathsf{Type}_{l+1} \\
\nu\mathsf{set}_l^{<0} & & \triangleq \ \mathsf{unit} \\
\nu\mathsf{set}_l^{<n'+1} & & \triangleq \ \Sigma X_{<n} : \nu\mathsf{set}_l^{<n'}.\, \nu\mathsf{set}_l^{=n}(X_{<n})) \\
\\
\nu\mathsf{set}_l^{=n} & (X_{<n} : \nu\mathsf{set}_l^{<n}) & : \ \mathsf{Type}_{l+1} \\
\nu\mathsf{set}_l^{=n} & X_{<n} & \triangleq \ \mathsf{fullframe}_l^n(X_{<n}) \to \mathsf{Type}_l
\end{array}
$$

where the "matching" $\mathsf{fullframe}_l^n$ is defined by mutual recursive construction (see later)

The recursive process used to build frames from layers of paintings

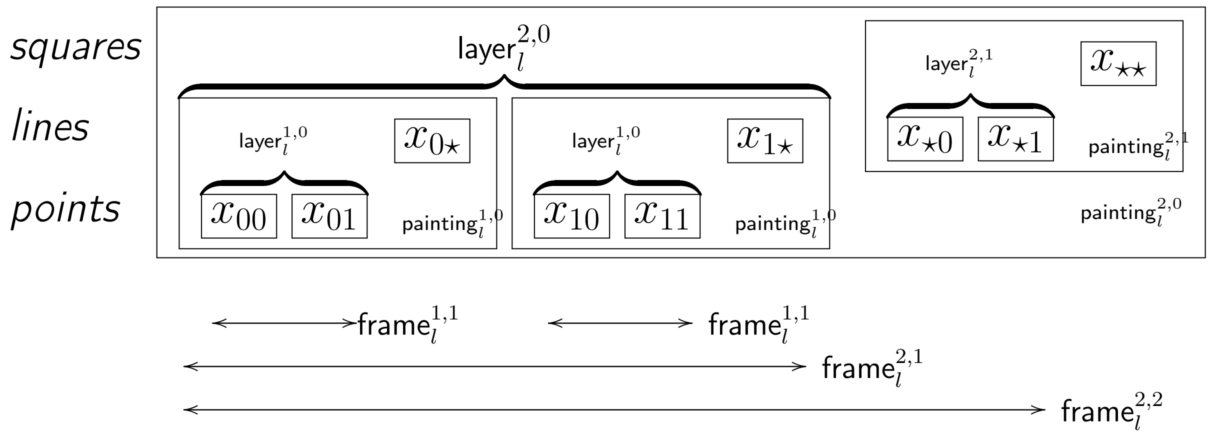


An n-frame is made of n layers, each made of two opposite paintings of decreasing intrinsic dimension and stretched to adjust to the dimension of the frame

The recursive construction, formally

$$
\begin{array}{lllll}
\mathsf{fullframe}^{n}_{l} & (X_{<n} : \nu\mathsf{set}^{<n}_{l}) & & : & \mathsf{Type}_l \\
\mathsf{fullframe}^{n}_{l} & X_{<n} & & \triangleq & \mathsf{frame}^{n,n}_{l}(X_{<n}) \\
\\
\mathsf{frame}^{n,p,[p\leq n]}_{l} & (X_{<n} : \nu\mathsf{set}^{<n}_{l}) & & : & \mathsf{Type}_l \\
\mathsf{frame}^{n,0}_{l} & X_{<n} & & \triangleq & \mathsf{unit} \\
\mathsf{frame}^{n,p'+1}_{l} & X_{<n} & & \triangleq & \Sigma d : \mathsf{frame}^{n,p'}_{l}(X_{<n}).\, \mathsf{layer}^{n,p'}_{l}(d) \\
\\
\mathsf{layer}^{n,p,[p<n]}_{l} & \{X_{<n} : \nu\mathsf{set}^{<n}_{l}\}\ (d : \mathsf{frame}^{n,p}_{l}(X_{<n})) & & : & \mathsf{Type}_l \\
\mathsf{layer}^{n,p}_{l} & (X_{<n-1}, X_{n-1})\ d & & \triangleq & \Pi\epsilon.\, \mathsf{painting}^{n-1,p}_{l}(X_{n-1})(\mathsf{restrframe}^{n,p}_{l,\epsilon}(d)) \\
\\
\mathsf{painting}^{n,p,[p\leq n]}_{l} & \{X_{<n} : \nu\mathsf{set}^{<n}_{l}\}\ (X_n : \nu\mathsf{set}^{=n}_{l}(X_{<n}))\ (d : \mathsf{frame}^{n,p}_{l}(X_{<n})) & & : & \mathsf{Type}_l \\
\mathsf{painting}^{n,p,[p=n]}_{l} & X_{<n}\ X_n\ d & & \triangleq & X_n(d) \\
\mathsf{painting}^{n,p,[p<n]}_{l} & X_{<n}\ X_n\ d & & \triangleq & \Sigma b : \mathsf{layer}^{n,p}_{l}(d).\, \mathsf{painting}^{n,p+1}_{l}(X_n)(d,b)
\end{array}
$$

which corresponds, when $\nu = 2$, to the following organisation of the 3^n components of a n-cube (shown for $n = 2$), with **frame**$_l$ associating layers on the left and **painting**$_l$ associating them on the right:



additionally, each atomic component at dimension n is a **painting**$^{n,n}_l$

The recursive construction: restrictions ("faces")

$$
\begin{array}{llcl}
\mathsf{restrframe}^{n,q,p,[p\leq q<n]}_{m,\epsilon} & \begin{array}{l}\{X_{<n} : \nu\mathsf{set}^{<n}_m\} \\ (d : \mathsf{frame}^{n,p}_m(X_{<n}))\end{array} & : & \mathsf{frame}^{n-1,p}_m(X_{<n-1}) \\
\mathsf{restrframe}^{n,q,0}_{m,\epsilon} & X_{<n}\ \star & \triangleq & \star \\
\mathsf{restrframe}^{n,q,p'+1}_{m,\epsilon} & X_{<n}\ (d,l) & \triangleq & (\mathsf{restrframe}^{n,q,p'}_{m,\epsilon}(d), \mathsf{restrlayer}^{n,q,p'}_{m,\epsilon}(l)) \\
\\
\mathsf{restrlayer}^{n,q,p,[p<q<n]}_{m,\epsilon} & \begin{array}{l}\{X_{<n} : \nu\mathsf{set}^{<n}_m\} \\ \{d : \mathsf{frame}^{n,p}_m(X_{<n})\} \\ (l : \mathsf{layer}^{n,p}_m(d))\end{array} & : & \mathsf{layer}^{n-1,p}_m(\mathsf{restrframe}^{n,q,p}_{m,\epsilon}(d)) \\
\mathsf{restrlayer}^{n,q,p}_{m,\epsilon} & (X_{<n-1}, X_{n-1})\ d\ l & \triangleq & \lambda\epsilon'.\,\overrightarrow{\mathsf{cohframe}^{n,p,q,p}_{m,\epsilon,\epsilon'}(d)}(\mathsf{restrpainting}^{n-1,q-1,p}_{m,\epsilon}(l_{\epsilon'})) \\
\\
\mathsf{restrpainting}^{n,q,p,[p\leq q<n]}_{m,\epsilon} & \begin{array}{l}\{X_{<n} : \nu\mathsf{set}^{<n}_m\} \\ \{X_n : \nu\mathsf{set}^{=n}_m(X_{<n})\} \\ \{d : \mathsf{frame}^{n,p}_m(X_{<n})\} \\ (c : \mathsf{painting}^{n,p}_m(X_n)(d))\end{array} & : & \mathsf{painting}^{n-1,p}_m(X_n)(\mathsf{restrframe}^{n,q,p}_{m,\epsilon}(d)) \\
\mathsf{restrpainting}^{n,q,p,[p=q]}_{m,\epsilon} & X_{<n}\ X_n\ d\ (l,\_) & \triangleq & l_\epsilon \\
\mathsf{restrpainting}^{n,q,p,[p<q]}_{m,\epsilon} & X_{<n}\ X_n\ d\ (l,c) & \triangleq & (\mathsf{restrlayer}^{n,q,p}_{m,\epsilon}(l), \mathsf{restrpainting}^{n,q,p+1}_{m,\epsilon}(c))
\end{array}
$$

where $\mathsf{cohframe}_{m,\epsilon,\epsilon'}$ is a coherence proof and we write $\overrightarrow{\mathsf{cohframe}_{m,\epsilon,\epsilon'}}$ for the rewriting of this proof from left to right

The recursive construction: coherences

$$
\begin{array}{lll}
\mathsf{cohframe}^{n,\,q,\,r,\,p}_{m,\epsilon,\epsilon'\;[p\,\leq\,r\,<\,q\,<\,n]} & \{X_{<n} : \nu\mathsf{set}^{<n}_m\} & : \ \mathsf{restrframe}^{n-1,q-1,p}_{m,\epsilon}(\mathsf{restrframe}^{n,r,p}_{m,\epsilon'}(d)) \\
 & (d : \mathsf{frame}^{n,p}_m(X_{<n})) & \quad = \mathsf{restrframe}^{n-1,r,p}_{m,\epsilon'}(\mathsf{restrframe}^{n,q,p}_{m,\epsilon}(d)) \\
\mathsf{cohframe}^{n,q,r,0}_{m,\epsilon,\epsilon'} & X_{<n}\ \star & \triangleq\ \mathsf{refl}\,\star \\
\mathsf{cohframe}^{n,q,r,p'+1}_{m,\epsilon,\epsilon'} & X_{<n}\ (d,l) & \triangleq\ (\mathsf{cohframe}^{n,q,r,p'}_{m,\epsilon,\epsilon'}(d), \mathsf{cohlayer}^{n,q,r,p'}_{m,\epsilon,\epsilon'}(l))
\end{array}
$$

$$
\begin{array}{lll}
 & \{X_{<n} : \nu\mathsf{set}^{<n}_m\{ & \\
\mathsf{cohlayer}^{n,\,q,\,r,\,p}_{m,\epsilon,\epsilon'\;[p\,<\,r\,<\,q\,<\,n]} & \{d : \mathsf{frame}^{n,p}_m(X_{<n})\} & : \ \mathsf{restrlayer}^{n-1,q-1,p}_{m,\epsilon}(\mathsf{restrlayer}^{n,r-1,p}_{m,\epsilon'}(l)) \\
 & (l : \mathsf{layer}^{n,p}_m(X_{<n})(d)) & \quad = \mathsf{restrlayer}^{n-1,r-1,p}_{m,\epsilon'}(\mathsf{restrlayer}^{n,q,p}_{m,\epsilon}(l)) \\
\mathsf{cohlayer}^{n,q,r,p}_{m,\epsilon,\epsilon'} & X_{<n}\ d\ l & \triangleq\ \lambda\epsilon''.\,\mathsf{cohpainting}^{n-1,q-1,r-1,p}_{m,\epsilon,\epsilon'}(l_{\epsilon''})
\end{array}
$$

$$
\begin{array}{lll}
 & \{X_{<n} : \nu\mathsf{set}^{<n}_m\} & \\
\mathsf{cohpainting}^{n,\,q,\,r,\,p}_{m,\epsilon,\epsilon'\;[p\,\leq\,r\,<\,q\,<\,n]} & \{X_n : \nu\mathsf{set}^{=n}_m(X_{<n})\} & : \ \mathsf{restrpainting}^{n-1,q-1,p}_{m,\epsilon}(\mathsf{restrpainting}^{n,r,p}_{m,\epsilon'}(c)) \\
 & \{d : \mathsf{frame}^{n,p}_m(X_{<n})\} & \quad = \mathsf{restrpainting}^{n-1,r,p}_{m,\epsilon'}(\mathsf{restrpainting}^{n,q,p}_{m,\epsilon}(c)) \\
 & (c : \mathsf{painting}^{n,p}_m(X_{<n})(X_n)(d)) & \\
\mathsf{cohpainting}^{n,q,r,p,[p=r]}_{m,\epsilon,\epsilon'} & X_{<n}\ X_n\ d\ (l,\_) & \triangleq\ \mathsf{refl}\,(\mathsf{restrpainting}^{n-1,q-1,p}_{m,\epsilon}(l_\epsilon)) \\
\mathsf{cohpainting}^{n,q,r,p,[p<r]}_{m,\epsilon,\epsilon'} & X_{<n}\ X_n\ d\ (l,c) & \triangleq\ (\mathsf{cohlayer}^{n,q,r,p}_{m,\epsilon,\epsilon'}(l),\ \mathsf{cohpainting}^{n,q,r,p+1}_{m,\epsilon,\epsilon'}(c))
\end{array}
$$

where we hide some coherences (such as proof-irrelevance of equality in **HSet** or the identification of the equality on pairs as a pair of equalities)

Adding reflexivities

We now set $\nu = 1$. For any $\nu\mathsf{set}_l$, we define a stream of reflexivities:

$$\begin{array}{l} \nu\mathsf{reflSet}(X_{-1}, X_0, ...) \quad \triangleq \\ \quad \Sigma r_{-1} : \Pi d : \mathsf{frame}^{-1}.\, \Pi x : X_{-1}(d).\, X_0(\mathsf{reflframe}^{-1}(d), x). \\ \quad \Sigma r_0 : \Pi d : \mathsf{frame}^{0}(X_{-1}).\, \Pi x : X_0(d).\, X_1(\mathsf{reflframe}^{0}(r_{-1})(d), x). \\ \quad \Sigma r_1 : \Pi d : \mathsf{frame}^{1}(X_{-1}, X_0).\, \Pi x : X_1(d).\, X_2(\mathsf{reflframe}^{1}(r_{-1}, r_0)(d), x). \\ \quad ... \end{array}$$

where

$$\mathsf{reflframe}^n(r_{-1}, ..., r_{n-1}) : \mathsf{frame}^n(X_{-1}, ..., X_{n-1}) \to \mathsf{frame}^{n+1,n}(X_{-1}, ..., X_n)$$

computes the n first layers of the border of $r_n(d)(x)$, knowing that the last layer is made of x itself, so that $(\mathsf{reflframe}^n(r_{-1}, ..., r_{n-1})(d), x)$ is a full frame (the matching map formerly called M_r), that is of type $\mathsf{frame}^{n+1}(X_{-1}, ..., X_n)$.

We also need two coherence conditions:

$$\begin{array}{l} \mathsf{idrestrreflframe}^n(r_{-1}, ..., r_{n-1}) : \Pi d : \mathsf{frame}^n.\, \mathsf{restrframe}^{n,n}_n(\mathsf{reflframe}^n(r_{-1}, ..., r_{n-1})(d)) = d \\ \mathsf{cohrestrreflframe}^n_{p<n}(r_{-1}, ..., r_{n-1}) : \Pi d : \mathsf{frame}^{n,p}. \\ \quad \mathsf{restrframe}^{n,p}_p(\mathsf{reflframe}^{n,p}(r_{-1}, ..., r_{n-1})(d)) = \mathsf{reflframe}^{n-1,p}(r_{-1}, ..., r_{n-2})(\mathsf{restrframe}^{n-1,p}_p(d)) \end{array}$$

where $\mathsf{reflframe}^{n,p}$ generalises $\mathsf{reflframe}^n$ to prefixes of frame^n:

$$\mathsf{reflframe}^{n,p}(r_{-1}, ..., r_{n-1}) : \mathsf{frame}^{n,p}(X_{-1}, ..., X_{n-1}) \to \mathsf{frame}^{n+1,p}(X_{-1}, ..., X_n)$$

Summary

- A work in progress, machine-checking a model following the iterated parametricity translation in indexed form.
- The addition of a reflexivity in the last direction is completed
- To be done: permutations, Π-types, Σ-types, universes, ...
- Also in progress: a more compact definition relying on finer-grain dependencies between the different components of the construction.