Progress Report on Constructive Higher Presheaf Models of HoTT

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Motivation

- We work in a *constructive* meta theory
  - Weak meta theory to make the results more generally applicable
  - Allows internalizing model construction into arbitrary models of ETT

- We want presheaf models for HoTT (over an internal category in **cSet**)

- Justification of Synthetic Approaches
  - Synthetic Algebraic Geometry (Cherubini, Coquand, and Hutzler, 2023)
  - Synthetic Stone Duality (Cherubini, Coquand, Geerligs, and Moeneclaey, 2024)
  - . . .

Why do we need an Internal Site?

Validity of Duality Axioms

- $k\text{-Alg}_{fp} \to \mathbf{cSet} \cong (\square^{\mathrm{op}} \times k\text{-Alg}_{fp}) \to \mathbf{Set}$ where $\square$ is some cube category
- The generic ring is $\mathrm{R}(A) :\equiv k\text{-Alg}_{fp}(k[X], A) \cong |A|$
- The duality axiom says that for each presentation[1] $(p_1, \ldots, p_m) \colon \mathrm{R}[X_1, \ldots, X_n]^m$

$$\frac{\mathrm{R}[X_1,\ldots,X_n]}{(p_1,\ldots,p_m)} \xrightarrow{\mathrm{ev}} \mathrm{R}^{(s\colon \mathrm{R}^n)\times\forall_i . p_i(s)\simeq 0}$$

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[1] given in a strict form, e.g, as a list of coefficients or trees

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$$\begin{tikzcd}
\frac{\mathrm{R}[X_1,\ldots,X_n]}{(p_1,\ldots,p_m)} \arrow[r, "\mathrm{ev}"] \arrow[dr, "\eta"'] & \mathrm{R}^{(s\colon \mathrm{R}^n)\times\forall_i . p_i(s)\simeq 0} & \\
& \frac{\mathrm{R}[X_1,\ldots,X_n]}{(p_1,\ldots,p_m)} \arrow[u, "\wr"'] & \text{strict}
\end{tikzcd}$$

- Since $\simeq$ in R is extensional equality, $\mathrm{R}^{(s\colon \mathrm{R}^n)\times\forall_i . p_i(s)\simeq 0}$ has no non-trivial paths.
- Strict axiom holds in general $\implies$ holds for HITs iff quotients are equivalent

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Internal Sites

- Resolve mismatch by passing to *internal site* of *cubical (0-truncated) k-algebras*

  $k\text{-Alg} \colon \square^{\mathrm{op}} \to \mathbf{Cat}$ “The type of 0-truncated k-algebras in $\mathrm{Psh}(\square)$”
- This category is fully defined in the language of HoTT
- We need a strict category for what follows, but path composition is not strict!

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- Use *relational* morphisms, i.e., phrase conditions for $h \colon A \to B$ as

$$(x, y, z \colon A) \to x + y \simeq_A z \to hx + hy \simeq_B hz$$

- For arbitrary $\mathbb{C}$ analogous construction: consider image of $よ \colon \mathbb{C} \hookrightarrow \mathrm{Psh}(\mathbb{C})$ and write natural transformations relationally $uf \simeq v \to (\alpha_I u)f \simeq \alpha_J(v)$

Internal Sites

- Category of internal presheaves on $\mathbb{C}\colon \square^{\mathsf{op}} \to \mathbf{Cat}$ is equivalent to $\mathsf{Psh}(\int_{\square} \mathbb{C})$
- In internal language of $\mathsf{Psh}(\square)$: just an ordinary presheaf category

$$\begin{tikzcd}
\mathsf{Psh}(\square) \arrow[r, shift left=2, "\pi^*"] \arrow[r, phantom, "\perp"] & \mathsf{Psh}(\int_{\square} \mathbb{C}) \arrow[l, shift left=2, "\pi_*"]
\end{tikzcd}$$

- The category $\mathsf{Psh}(\int_{\square} \mathbb{C})$ is a setting for the cubical model construction
  - tiny interval $(\pi^*\mathbb{I})(I, X) = \mathbb{I}(I)$
  - cofibration classifier $(\pi^*\Phi)(I, X) = \Phi(I)$
- We obtain a model of HoTT with HITs

Working with Strictified Internal Sites

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  - These will in general only be fibrant *levelwise*

- Other objects are more problematic, e.g., quotients of and free algebras over R
  - These can be defined levelwise as an HIT
  - Are only presheaves *up to homotopy*

$$\begin{tikzcd}
A \arrow[dd, "fg"'] \arrow[dr, "g"] & & & A[X] \arrow[dd, "\overline{fg}"'] \arrow[dr, "\overline{g}"] & \\
& B \arrow[dl, "f"] & \longmapsto & {} & B[X] \arrow[dl, "\overline{f}"] \arrow[l, Rightarrow] \\
C & & & C[X] &
\end{tikzcd}$$

Strictification Modality (Coquand, Ruch, and Sattler, 2021)

- Consider external categories, but these results can be generalized to our setting
- Introduce modality $\underline{E}$ s.t. $A \xrightarrow{\text{weak}} B$ correspond to $A \xrightarrow{\text{strict}} \underline{E}B$

Lemma (Levelwise Principle)

For an $\underline{E}$-modal type A, we have $\mathsf{El}_{\mathsf{Psh}(\int \mathbb{C})}(\Gamma, \|A\|) \longleftrightarrow \mathsf{El}_{\mathsf{Psh}(\square)}(\mathbb{C}_0.\Gamma, \|A\|)$.

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- The construction can be factored over a notion of weak presheaf
- Allows working with weak objects needed for SAG and SSD settings

$$\begin{tikzcd} \mathrm{Psh}_w(\int \Gamma) \arrow[r, shift left=2, "R_\Gamma"] \arrow[r, phantom, "\text{``}\top\text{''}"] & \mathrm{Psh}(\int \Gamma) \arrow[l, shift left=2, "U_\Gamma"] \arrow[loop right, "\underline{E}"] \end{tikzcd}$$

$(\text{Nat in } \Gamma :\ \mathrm{Psh}(\mathbb{C}))$

Summary until now

- To obtain the duality axioms, we need to pass to internal sites
- To deal with the resulting coherence issues, we need the modality $\underline{E}$

- For axioms of *synthetic stone duality*, we also need dependent choice

Sattler's Model of ∞-Groupoids (2023)

- Let $\boxbar \coloneqq \mathbf{Pos}_{\mathsf{Fin},\neq\emptyset}$ and $\boxbslash \colon \ \mathsf{Psh}(\boxbar) \xrightarrow{i^*} \mathsf{Psh}(\Delta_+) \xrightarrow{i_*} \mathsf{Psh}(\boxbar)$
- Defining property $\mathsf{El}_{\mathsf{Psh}(\boxbar)}(\Gamma, \boxbslash A) \cong \mathsf{El}_{\mathsf{Psh}(\Delta_+)}(i^*\Gamma, i^*A)$

- We take the submodel of modal types for $\boxbslash$
- Associated model structure for this model is Quillen equivalent to the (constructive) Kan model structure on $\mathsf{Psh}(\Delta)$

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Lemma (Pointwise Principle)

For $\boxbslash$-modal types, there is a logical equivalence $\mathsf{El}_{\mathsf{Psh}(\square)}(\Gamma, \|A\|) \longleftrightarrow \mathsf{El}_{\mathbf{Set}}(|\Gamma|, |A|)$.

- As a consequence, dependent choice holds in the model
- The presheaf construction should preserve this property

Cubical Presheaves over Internal Categories

Putting it all Together

Lemma

The pointwise lifted $\boxbslash$ *modality preserves fibrant types in the sense of* $\mathrm{Psh}(\int \mathbb{C})$*.*

To combine the two modalites and obtain both principles we need the following.

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The modality $\underline{E}$ *preserves* $\boxbslash$*-modal types.*

We cannot yet make the conclusion below. The underlying family of types of an internal presheaf might not be fibrant in the sense of $\mathsf{Psh}(\square)$.

$$\begin{tikzcd}
\mathsf{El}_{\mathsf{Psh}(\int \mathbb{C})}(\Gamma, \|A\|) \arrow[r, leftrightarrow, "\underline{E}"] & \mathsf{El}_{\mathsf{Psh}(\square)}(\mathbb{C}_0.\Gamma, \|A\|) \arrow[r, leftrightarrow, "\boxbslash"] & \mathsf{El}_{\mathbf{Set}}(|\mathbb{C}_0|.|\Gamma|, |A|)
\end{tikzcd}$$

Cubical Presheaves over Internal Categories

Fibrant Categories

Issue $A \in \mathrm{Ty}_{\mathrm{Psh}(\int \mathbb{C})}(\Gamma)$ fibrant $\not\Rightarrow$ associated $A \in \mathrm{Ty}_{\mathrm{Psh}(\square)}(\mathbb{C}_0.\Gamma, A)$ fibrant

Definition

A cubical category $\mathbb{C} \colon \square^{\mathrm{op}} \to \mathbf{Cat}$ is *fibrant* if $\mathbb{C}_1 \to \mathbb{C}_0 \times \mathbb{C}_0$ is a fibration.

- Equivalently, $\mathbb{C}_1 \colon \mathbb{C}_0 \times \mathbb{C}_0 \to \mathrm{U}$ family of fibrant types
- Given a path $x \colon x_0 \simeq_{\mathbb{C}_0} x_1$ we can built a line $f \colon (i \colon \mathbb{I}) \to \mathbb{C}_1(x_0, x_i)$ with $f_0 = \mathrm{id}_{x_0}$

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Lemma

Let $\mathbb{C} \colon \square^{\mathrm{op}} \to \mathbf{Cat}$ *be a fibrant category.*
If $A \in \mathrm{Ty}_{\mathrm{Psh}(\int \mathbb{C})}(\Gamma)$ *fibrant then the associated* $A \in \mathrm{Ty}_{\mathrm{Psh}(\square)}(\mathbb{C}_0.\Gamma, A)$ *is fibrant.*

Choice Axioms for Presheaf Models

Theorem

For a fibrant cubical category $\mathbb{C} \colon \square^{\mathrm{op}} \to \mathbf{Cat}$, *the operation* $\underline{E} \circ \boxbslash$ *is a lex modality on* $\mathrm{Psh}(\int_{\square} \mathbb{C})$, *and the submodel of modal types is*

1. *a model of* HoTT *(with HITs),*
2. *with a logical equivalence* $\mathrm{El}_{\mathrm{Psh}(\int_{\square} \mathbb{C})}(\Gamma, \|A\|) \leftrightarrow \mathrm{El}_{\mathbf{Set}}(|\mathbb{C}_0|.|\Gamma|, |A|)$ *natural in* Γ.

- General tool for constructing presheaves models
- All our categories of interest satisfy the fibrancy condition
- The pointwise principle allows us to conclude dependent choice for SSD

Conclusion and Future Work

- Finish specific applications (e.g. SAG, SSD, . . .)
- Compare cubical with other models of higher presheaves
- Formalize existing construction
  - large parts can be done in internal language of $\mathrm{Psh}(\square)$
  - especially the technical verification of axioms for the applications

Dependent Choice Axiom from Pointwise Principle

- $\mathsf{El}_{\mathsf{Psh}(\int \mathbb{C})}(\Gamma, \|A\|) \leftrightarrow \mathsf{El}_{\mathbf{Set}}(|\mathbb{C}_0|.|\Gamma|, |A|)$ implies dependent choice

$$\mathsf{El}_{\mathsf{Psh}(\int \Gamma)}(\Gamma, \|A_0\|) \longrightarrow \mathsf{El}_{\mathbf{Set}}(|\mathbb{C}_0|.|\Gamma|, |A_0|)$$

$$\mathsf{El}_{\mathsf{Psh}(\int \Gamma)}(\Gamma, \Pi_{n :\, \mathsf{N}, a_n :\, A_n} \|\mathsf{Fib}(f_n, a_n)\|) \longrightarrow \mathsf{El}_{\mathbf{Set}}(|\mathbb{C}_0|.|\Gamma|, \Pi_{n:\mathbb{N}, a_n:|A_n|} \Sigma_{a_{n+1}:|A_{n+1}|} |f_n a_{n+1} \simeq a_n|)$$

- Then we can argue using induction in **Set** and conclude

$$\mathsf{El}_{\mathsf{Psh}(\int \Gamma)}(\Gamma, \|\Sigma_{u :\, \Pi_{\mathsf{N}} A} \Pi_{i :\, \mathsf{N}} f_i u_{i+1} \simeq u_i\|) \longleftarrow \mathsf{El}_{\mathbf{Set}}(|\mathbb{C}_0|.|\Gamma|, \Sigma_{u :\, \Pi_{\mathbb{N}} |A|} \Pi_{i :\, \mathbb{N}} |f_i u_{i+1} \simeq u_i|)$$

Cubical Presheaves over Internal Categories

Setting $\square^{\mathrm{op}} : \mathbb{C} \to \mathbf{Cat}$, a strict (closed) category in Psh($\square$)

- The category $\mathrm{Psh}(\int_\square \mathbb{C})$ has
  - tiny interval $(\pi^*\mathbb{I})(I, X) = \mathbb{I}(I)$
  - universal cofibration $\pi^*\top : \pi^*1 \to \pi^*\Phi$
- Cubical model construction applies

Problem

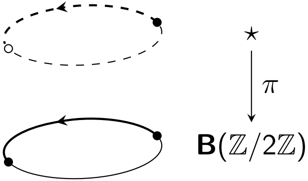
- If $\mathbb{C} = \mathbb{C}_0$ the model is not the slice model!
- This is not the same issue as in CRS since it occurs for $\mathbb{C}_0$
- There are non-trivial paths in the type of objects

$$\begin{tikzcd}
\mathrm{Psh}(\square) \downarrow \mathbb{C}_0 \\
\mathrm{Psh}(\int_\square \mathbb{C}_0) \arrow[u, phantom, "\wr"] \arrow[d, shift left=3, "i_*"] \arrow[d, phantom, "\dashv"] \\
\mathrm{Psh}(\int_\square \mathbb{C}) \arrow[u, shift left=3, "i^*"] \arrow[d, shift left=3, "\pi_*"] \arrow[d, phantom, "\dashv"] \\
\mathrm{Psh}(\square) \arrow[u, shift left=3, "\pi^*"]
\end{tikzcd}$$

Cubical Presheaves over Internal Categories

Notions of Fibrancy

- Comparison of fibrancy in case of discrete category
- Consider the family $(- = \star)$ over $\mathbf{B}(\mathbb{Z}/2\mathbb{Z})$
  - Fibrant in $\mathsf{Psh}(\int \mathbf{B}(\mathbb{Z}/2\mathbb{Z}))$ constructed with $\pi^*\mathbb{I}, \pi^*\Phi$
  - Modal for ⧅-modality (lifted in the obvious way)
  - Inhabited on points since $\star = \star$
  - not inhabited, $\mathsf{El}_{\mathsf{Psh}(\int \mathbf{B}(\mathbb{Z}/2\mathbb{Z}))}(1, (- = \star)) = \emptyset$
- This type is not fibrant in the slice



Fibrancy

- An internal (dependent) presheaf is fibrant if $A_0 \to \Gamma_0$ is a fibrantion, and the restriction action (the square on the left) is a morphism of fibrations

$$\begin{tikzcd}
\mathbb{C}_0.\Gamma_0.A_0 \arrow[d, two heads] \arrow[dr, phantom, "Mor"] & \mathbb{C}_1.\Gamma_1.A_1 \arrow[l] \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{C}_0.\Gamma_0.A_0 \arrow[d, two heads] \\
\mathbb{C}_0.\Gamma_0 & \mathbb{C}_1.\Gamma_1 \arrow[l] \arrow[r] & \mathbb{C}_0.\Gamma_0
\end{tikzcd}$$

- In the model we obtain from the construction, these will not be fibrations in the sense of Psh($\square$)
- Instead, we only consider those lifting problems where the path is constant on $\mathbb{C}_0$

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