The Yoneda embedding in simplicial type theory

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What are we trying to do?

Two things which are neat:[1]

1. (Homotopical) dependent type theory,
2. (Homotopical) category theory.

Our goal

Let's make these two independently neat things even better by combining them.

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More specifically:

Our goal (revised)

We want a HoTT-variant tuned to prove facts about $(\infty, 1)$-categories.

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The complication: categories are not groupoids

The fundamental challenge: variance.

- Cat (∞ or not) is not locally Cartesian closed.
- Consequently, not every operation in HoTT can be interpreted by Cat.
- Also *new* operations (e.g., $-^{\mathsf{op}}$) which have no clear counterpart.

Two solutions:

1. Change type theory a lot but work with basically Cat
2. Change type theory a little but work with something more complex than Cat

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See e.g., [War13; Nuy15; Kav19; KS23; ANW23; Nor18; Nuy20; NA24] for (1).

The solution: moving the goal posts

Big Idea

Following [RS17]:

- Embed Cat fully-faithfully into the (∞-)topos $\mathrm{PSh}(\Delta)$.
- Use ordinary HoTT + axioms to reason about the image of this embedding (isolated by 2 propositions!).
- Throw in a handful of modalities for good measure

(Axioms are closely related those of Synthetic Algebraic Geometry [CCH23])

Simplicial Type Theory

Simplicial type theory extends ordinary HoTT to reason about $\mathrm{PSh}(\Delta)$:

- 4 modalities (à la MTT [Gra+20]) including $\mathfrak{o}$ and $\flat$,
- $\sim$8 axioms postulating a synthetic directed interval $\mathbb{I}$ and controlling its behavior.

Using $\mathbb{I}$, we define commuting triangles, composable pairs, and isomorphisms:

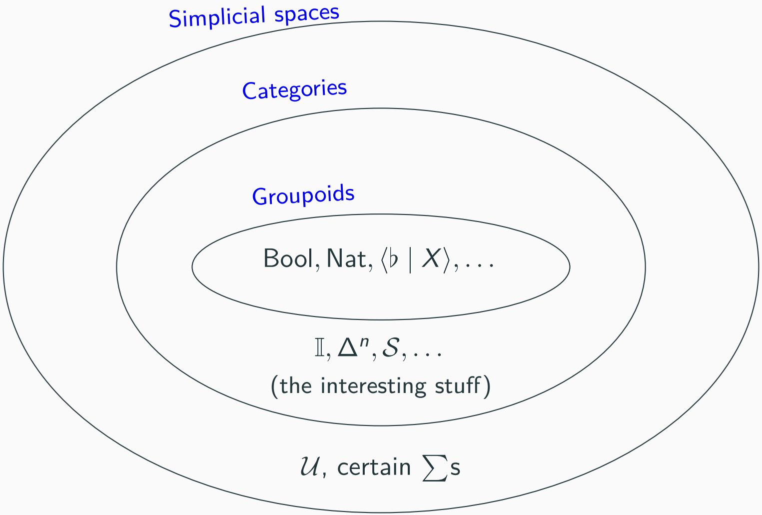
$$\Delta^2 = \begin{array}{ccc} \bullet & \longrightarrow & \bullet \\ & \searrow & \downarrow \\ & & \bullet \end{array} \qquad \Lambda^2_1 = \begin{array}{ccc} \bullet & \longrightarrow & \bullet \\ & & \downarrow \\ & & \bullet \end{array} \qquad E = \bullet \;\substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow}\; \bullet$$

Not every type is a category, but every category is a type:

$$\hom_C(c, d) = \textstyle\sum_{f : \mathbb{I} \to C} f\,0 = c \times f\,1 = d$$

$$\mathrm{isCategory}(C) = \mathrm{isEquiv}(C^{\Delta^2} \to C^{\Lambda^2_1}) \times \mathrm{isEquiv}(C \to C^E)$$

A map of $\mathsf{PSh}(\Delta)$



Our contribution

Prior work

- [RS17; Bar22]: limits, colimits, adjunctions, etc. are definable in STT.
- [GWB24]: there is a directed-univalent category $\mathcal{S} \subseteq \mathcal{U}$ of groupoids.

We put these two together and show many of the foundational results are within reach:

- There is a fully-faithful functor $C \to \mathrm{PSh}(C) = \mathcal{S}^{\langle \mathfrak{o} | C \rangle}$.
- $\mathrm{PSh}(C)$ is (co)complete (in fact, the free cocompletion of C).
- Existence of and formula for pointwise Kan extensions
- Quillen's theorem A
- ...

The big idea

The proofs of many of these results boil down to the same maneuver:

- Directed univalence tells us that χ holds of $\mathcal{S}$.
- χ is checked pointwise in $\mathsf{PSh}(C)$, so it holds for all presheaf categories.
- Since C is full subcategory of $\mathsf{PSh}(C)$, χ holds for C as well.

The third step is the main technical contribution:

Linchpin construction

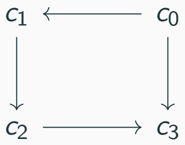
$\Phi = \hat{\mathbf{y}} : \langle \mathfrak{o} \mid C \rangle \times C \to \mathcal{S}$ which defines a functorial version of $\hom(-,-)$.

From twisted arrows to Φ

A few words about Φ are in order. The most important are "twisted arrow".

- We realize the total space of Φ as a modality $\langle \mathfrak{t} \mid C \rangle$.
- The modal machinery gives us the required projection functions.
- We add an axiom ensuring $\langle \mathfrak{t} \mid - \rangle$ represents the right functor.

Objects of $\langle \mathfrak{t} \mid C \rangle$ are arrows in C and arrows are "twisted" squares:



The geometry of Φ

To show that Φ factors through $\mathcal{S}$, we must show that it is covariant:

Theorem

If $A :_\flat X \to \mathcal{U}$ then A factors through $\mathcal{S}$ if and only if $\sum_{x:X} A(x) \to X$ is right orthogonal to $\{0\} \to \Delta^n$ for all n.

So we need an inverse to

$$\langle \flat \mid \langle \mathfrak{t} \mid C \rangle^{\Delta^n} \rangle \to \langle \flat \mid \langle \mathfrak{t} \mid C \rangle \rangle \times_{\langle \flat \mid \langle \mathfrak{o} \mid C \rangle \times C \rangle} \langle \flat \mid (\langle \mathfrak{o} \mid C \rangle \times C)^{\Delta^n} \rangle$$

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Roughly, this inverse sends:

$$\left(\begin{array}{ccccc} c_n & \longrightarrow & c_{n+1}, & & \\ c_n & \longleftarrow & c_{n-1} & \longleftarrow & \cdots, \\ c_{n+1} & \longrightarrow & c_{n+2} & \longrightarrow & \cdots \end{array} \right) \Longrightarrow \begin{array}{ccccccc} c_n & \longleftarrow & c_{n-1} & \longleftarrow & \cdots & \longleftarrow & c_0 \\ \downarrow & & & & & & \\ c_{n+1} & \longrightarrow & c_{n+2} & \longrightarrow & \cdots & \longrightarrow & c_{2n} \end{array}$$

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Think discretely, act functorially

With **y**, we can prove a number of useful rectification results which amount to the following for various values of Ψ:

Theorem

If there is a function $(c :_\flat C) \to \Psi(c)$ *then there is a unique function* $(c : C) \to \Psi(c)$

For instance:

- $\Psi(c) =$ "$\alpha_c : \hom(F\,c, G\,c)$ is an isomorphism" (pointwise invertible = invertible)
- $\Psi(c) =$ "$\Phi(L(c), -)$ is representable".

From the last one, we upgrade a "pointwise" adjoint to a full adjunction.

A potpourri of adjunctions

Let's read off some adjunctions:

- $f^* : \mathcal{S}^D \to \mathcal{S}^C$ with $f : C \to D$ is both left and right adjoint.
- $i : \mathcal{S}_{\leq n} \to \mathcal{S}$ is a right adjoint.
- $\Omega^\infty : \mathrm{Sp} \to \mathcal{S}_*$ is a right adjoint.
- $i : \Delta^{\mathrm{inj}} \to \Delta$ is a right adjoint.

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- $i : \Delta^{\mathrm{inj}} \to \Delta$ is a right adjoint.
- $\Phi(f(-), -) : E \to \mathrm{PSh}(C)$ is a right adjoint if E is cocomplete.

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Corollary

If E is cocomplete then $\mathrm{isEquiv}(\mathbf{y}^* : (\mathrm{PSh}(C) \to^L E) \to (C \to E))$

Kan extensions

The adjoints to $f^* : \mathcal{S}^D \to \mathcal{S}^C$ are given by left/right Kan extension...

Theorem

If E is (co)complete then $E^D \to E^C$ has a (left) right adjoint.

- Proof reduces to the case of $\mathrm{PSh}(-)$ where it follows from above.
- One can unravel to recover the usual formula for pointwise Kan extensions.

Quillen's theorem A

With Kan extensions to hand, we can prove the following:

Theorem (Quillen)

If $f :_\flat C \to D$ then the following are equivalent:

- *f is right cofinal.*
- *For all $d :_\flat D$ the category $C \times_D D_{d/}$ is weakly contractible.*
- $\varinjlim_D \circ f^* = \varinjlim_C : D \to \mathcal{S}$.

Why does this matter? In practice, we make C very simple and D is quite complex.

A little example: cocartesian maps are proper

Theorem

If $\pi :_\flat E \to B$ is cocartesian & $u :_\flat A \to B$ is right cofinal then $\pi^* u : A^* E \to E$ is right cofinal.

By Theorem A, we need to prove $\bigcirc_{\mathsf{grpd}}(A \times_B E_{e/}) \simeq \mathbf{1}$. for this we calculate...

A little example: continued

$$\bigcirc_{\mathsf{grpd}}(A \times_B E_{e/})$$

(π is cocartesian)

$$\simeq \bigcirc_{\mathsf{grpd}}\left(A \times_B \left(\sum_{b' : B, f : \hom(\pi(e), b')} (E_{b'})^{\mathbb{I}}\right)\right)$$

(Reshuffling some Σs and equalities)

$$\simeq \bigcirc_{\mathsf{grpd}}\left(\sum_{(a,f) : A_{\pi(e)/}} (E_{u(a)})_{f_! e/}\right)$$

(Distribute $\bigcirc_{\mathsf{grpd}}$ to each fiber)

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(Coslices have initial objects)

$$\simeq \bigcirc_{\mathsf{grpd}}(A_{\pi(e)/})$$

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$$\simeq \mathbf{1}$$ $\square$

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□

Conclusions

With just the Yoneda embedding, quite a lot of category theory is within reach!

- We can develop the theory of presheaf categories
- From this, Kan extensions, cofinality, etc.
- Proofs are (subjectively) not horrible
- `https://arxiv.org/abs/2501.13229` (this paper)
- `https://arxiv.org/abs/2407.09146` (the $\mathcal{S}$ paper)

Stay tuned! Filtered colimits, stable homotopy theory, animation, etc.

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