# Axiomatization of $\infty$-categories

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*Joint work with Denis-Charles Cisinski, Kim Nguyen and Tashi Walde*

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# Goal: Axiomatizing Higher Category Theory

**Overview:**

- This talk presents an ongoing project aimed at developing foundations for higher category theory.
- We seek a user-friendly meta theory capturing essential aspects used in practice.
- Our approach is deliberately 'substrate-agnostic' initially.
- **Warning:** Axioms and the underlying 'substrate' are still evolving.

*Everything presented is joint work with Denis-Charles Cisinski, Kim Nguyen and Tashi Walde.*

# What are $\infty$-categories?

**Slogan:** $\infty$-category theory is **homotopical category theory**:

- Like category theory, but where all strict equalities are replaced by homotopical equalities.
- Like HoTT, but where types have *hom types* $\hom_{A(x,y)}$ functioning as *directed* equality types.

$\infty$-categories are becoming an essential tool across mathematics: in algebraic topology, algebraic geometry, representation theory, etcetera.

# Why axiomatize $\infty$-categories? (1/2)

**Motivation 1**: Resolve mismatch foundations vs. practice

- Current standard: Set theory (quasicategories).
- **Issue:** Set theory's strictness clashes with the homotopical nature of $\infty$-categories.
- Leads to high technical overhead, often disconnected from practical usage.
- **Want:** Foundations that match practice.

Design criteria:

- Easy to learn/use, fitting with intuition;
- Capturing more general (e.g. '*parametrized*') category theories.

# Why axiomatize $\infty$-categories? (2/2)

**Motivation 2:** Provide a theory that can "eat itself"

- The 'universe' $\mathrm{Cat}$ of $\infty$-categories is itself an $\infty$-category;
- All axioms should be satisfied internally to $\mathrm{Cat}$;
- *Key difference with HoTT*: The universe can natively talk about *maps* of types;
- No problem defining infinite structures, e.g. simplicial types can be defined as functors $X \colon \Delta^{\mathrm{op}} \to \mathrm{Cat}$.

# Two key questions

**Key Question 1 (this talk)**

What is a suitable **meta language** for higher category theory? What are its **core principles**?

**Key Question 2 (future)**

What is a suitable **substrate** for higher category theory?

**Set Theory**

- Quasicategories
- Complete Segal Spaces

**Category Theory**

- $\infty$-cosmoi
- Tribes

**Type Theory**

- Simplicial TT
- Directed TT (?)
- Globular Types
- Plain Agda
- ...?

# Meta versus substrate

**Meta language**

- Categories are primitive notion
- Universal properties stated 'by hand'
- Coherences stated 'by hand'

**Tribes**

- Categories are objects of a tribe $\mathcal{E}$
- Categorical universal properties
- Coherences deducible

**Simplicial TT**

- Categories are types
- Universal properties via inference rules
- Coherences deducible

# The resulting theory

Within our framework, we can develop:

- Adjunctions, limits/colimits, Kan extensions;
- Localizations;
- Cofinality, Quillen's theorems A and B;
- Yoneda embedding;
- Peano arithmetics;
- (Co)cartesian fibrations, Straightening/unstraightening;
- "All of HoTT" (the theory of groupoids).

In the future, we hope to extend this to:

- Presentable categories, topoi;
- Stable homotopy theory;
- Higher algebra, operads;
- ...

# The primitives

We take the following as primitive:

- Synthetic categories: $C, D, E, \ldots$
- Functors: $f\colon C \to D$
- Natural **isomorphisms**: $\alpha\colon f \cong g$
- Higher natural isomorphisms (ad infinitum)
- Basic operations (composition, identity) satisfying coherence laws (associativity, unitality up to higher iso)

**Key Point:** Only *invertible* transformations are primitive. Think of $\alpha$ as a homotopy $f \simeq g$ in HoTT.

# Category constructors

We need standard ways to construct categories:

- Initial / Terminal categories: $\emptyset, *$
- Products / Coproducts: $C \times D, \quad C \sqcup D$
- Pullbacks: $C \times_E D$
- Functor categories: $\mathrm{Fun}(C, D)$

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They come with their 'naive' (non-coherent) universal properties, e.g.:

- For every $C$ we get $p_C \colon C \to *$, unique up to iso;
- Pairs $(f : E \to C, g : E \to D)$ correspond to $(f, g) \colon E \to C \times D$;
- Functors $E \to \mathrm{Fun}(C, D)$ correspond to functors $E \times C \to D$.

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This is enough: we can prove things like

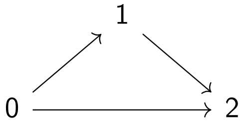
$$\mathrm{Fun}(E, C \times D) \simeq \mathrm{Fun}(E, C) \times \mathrm{Fun}(E, D).$$

# Morphisms and Composition

To talk about directionality (morphisms, diagrams), we need standard shapes:

- The "walking morphism": $[1] = (0 \to 1)$
  - A *morphism* $f : x \to y$ in $C$ is a functor $f : [1] \to C$ with $f(0) \cong x$, $f(1) \cong y$.
  - (An *object* is a functor $x : * \to C$).
- The "walking commutative triangle": $[2] = (0 \to 1 \to 2)$

$[2] \quad =$




We also require standard maps $d_i : [1] \to [2]$ and $s_j : [2] \to [1]$ satisfying the simplicial identities (finite list of conditions).

# Diagram axioms (1/3)

**Commutative square axiom:** For any category $C$, the restriction map is an equivalence:

$$\mathrm{Fun}([1] \times [1], C) \xrightarrow{\simeq} \mathrm{Fun}([2], C) \times_{\mathrm{Fun}([1], C)} \mathrm{Fun}([2], C)$$

$$\begin{array}{ccc} x & \longrightarrow & y \\ \downarrow & & \downarrow \\ z & \longrightarrow & w \end{array} \qquad \longleftrightarrow \qquad \begin{array}{ccc} x & \longrightarrow & y \\ \downarrow & \searrow & \downarrow \\ z & \longrightarrow & w \end{array}$$

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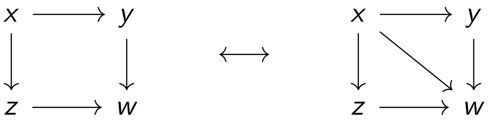


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**Informally:** Categories understand how squares relate to triangles.

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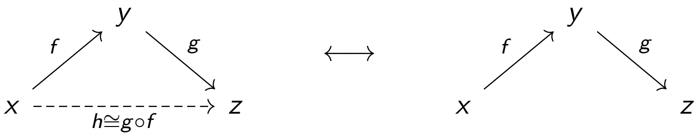


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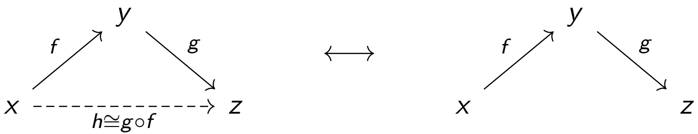


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Then the zigzag

$$\mathrm{Fun}([1],C) \times_C \mathrm{Fun}([1],C) \xleftarrow{\sim} \mathrm{Fun}([2],C) \xrightarrow{d_1} \mathrm{Fun}([1],C)$$

gives composition $-\circ-\colon \mathrm{Hom}_C(y,z) \times \mathrm{Hom}_C(x,y) \to \mathrm{Hom}_C(x,z)$.

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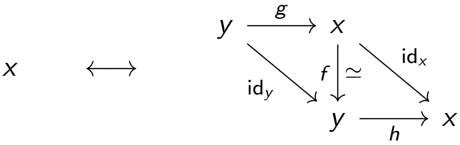
$$C \xrightarrow{\simeq} \mathrm{Iso}(C), \qquad x \mapsto \mathrm{id}_x$$

is an equivalence. Here $\mathrm{Iso}(C)$ is the category whose objects are isomorphisms $f\colon x \to y$ (with data $g\colon y \to x$, $fg \cong \mathrm{id}_y$, $h\colon y \to x$ $hf \cong \mathrm{id}_x$).



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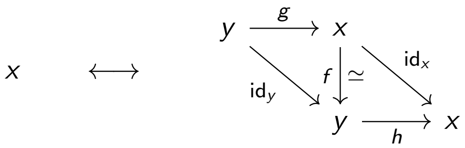


**Informally:** Categories treat isomorphic objects as equal.

# Overview basic language

So far, we have

1. Primitive notions (categories, functors, natural isos, ...)
2. Basic constructors ((co)products, pullbacks, functor categories)
3. The posets [1] and [2] ($\rightsquigarrow$ morphisms, commutative triangles)
4. Diagram axioms ($\rightsquigarrow$ composition of morphisms)

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With this, one can already develop some basic theory:

- Adjunctions;
- Initial/terminal objects;
- Slice categories;
- Cartesian and cocartesian fibrations.

(Formalized in simplicial type theory by Riehl–Shulman and Buchholtz–Weinberger.)

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3. **Localizations:** Given $W \hookrightarrow \mathrm{Map}([1], C)$, define $C \to C[W^{-1}]$, universal functor inverting morphisms in $W$.

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5. **Joins:** $C \star D$, fitting into a pushout square:

$$\begin{array}{ccc} C \times D \sqcup C \times D & \longrightarrow & C \times [1] \times D \\ \downarrow & & \downarrow \\ C \sqcup D & \longrightarrow & C \star D \end{array}$$

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1. **Functoriality of Universals:**
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2. **Exponentiability Axiom:**
   - All (co)cartesian fibrations $p\colon E \to C$ are exponentiable.
   - Meaning: Dependent product along $p$ exists. The pullback functor $p^*\colon \mathrm{Cat}_{/C} \to \mathrm{Cat}_{/E}$ has a right adjoint $p_*$ (satisfying Beck-Chevalley).

   *Important*: Not all dependent products exist! (Leads to subtleties for formalization.)

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   - "Universe": $U$ is closed under the category constructors.
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     $$\mathrm{Hom}_U(C, D) \to \mathrm{Map}(C, D)$$
     is an equivalence.
   - When restricted to the equivalences, this recovers ordinary univalence.

# Recap

**Goal:** Develop a meta-language for higher category theory.

- Approach: Build up theory from primitives, constructors, and key axioms.
- Focus on core principles that mirror practical usage.
- Resulting theory interpretable in various foundational substrates (STT, tribes, etc.).
- **Future wish:** Type theory in which *all* types are categories.

Our book-project may be found on my homepage:
`https://sites.google.com/view/bastiaan-cnossen`