The algebraic small object argument as a saturation

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joint work with Christian Sattler

Story

- ⊗ Writing up argument that many cubical-style model structures don't present spaces...
- ⊗ Funny but central lemma:

$$A \overset{\sim}{\rightarrowtail} B \quad \Longrightarrow \quad A^{\square^n/\sigma} \overset{\sim}{\rightarrowtail} B^{\square^n/\sigma}$$

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  for well-chosen n and σ, where $\overset{\sim}{\rightarrowtail}$ = trivial cofibrations
- $\otimes$ Usual strategy:
  - $\oslash$ Check for generating $\overset{\sim}{\rightarrowtail}$ ($\approx$ open box inclusions)
  - $\oslash$ Extend to all $\overset{\sim}{\rightarrowtail}$ by one of two routes:
    - $\odot$ If $(-)^{\square^n/\sigma}$ is left adjoint, done (no dice!)
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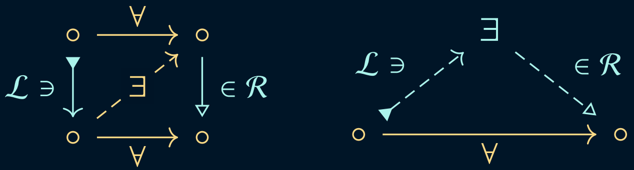
Theme: let's get structured!

Weak factorization systems

- On a category $\mathscr{E}$, pair $(\mathcal{L}, \mathcal{R})$ of $\mathcal{L}, \mathcal{R} \subseteq \mathrm{Ob}\,\mathscr{E}^{\rightarrow}$



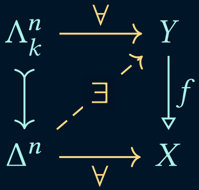
- Examples:
  - (complemented mono, split epi) in adhesive categories
  - (trivial cofibration, Kan fibration) in simplicial sets
  - (trivial cofibration, uniform Kan fibration) in cubical sets

Generation by a set

- A WFS $(\mathcal{L}, \mathcal{R})$ might be generated by a set $S \subseteq \mathrm{Ob}\,\mathscr{E}^{\to}$:

$$\mathcal{R} = \{f \in \mathscr{E}^{\to} \mid f \text{ right lifts against all } m \in S\}$$

- Examples:
  - (complemented mono, split epi) in **Set** generated by $\{0 \rightarrowtail 1\}$
  - (trivial cofibration, Kan fibration) in simplicial sets generated by $\{\Lambda^n_k \rightarrowtail \Delta^n \mid n \leq k \in \mathbb{N}\}$



Generation by a category

- f right lifts against $u : \mathscr{J} \to \mathscr{E}^{\to}$ when

$\forall j \in \mathscr{J}$ $\qquad\qquad$ $\forall \alpha : i \to j$

- Examples:
  - (complemented mono, split epi) in **AbGrp** generated by full subcat of **CompMono**(**AbGrp**) on $0 \overset{!}{\rightarrowtail} \mathbb{Z}$ & $\mathbb{Z} \overset{\Delta}{\rightarrowtail} \mathbb{Z} \times \mathbb{Z}$
  - (trivial cofibration, uniform Kan fibration) in cubical sets

Generation by a category: uniform fibrations

- Generated by a diagram $u\colon \{0,1\} \times \Phi \longrightarrow \mathrm{PSh}(\square)^{\to}$, where Φ a subcategory of monos $A \rightarrowtail \square^n$ and pullback squares between them

$$k \in \{0,1\}, \quad \begin{array}{c} A \\ {\scriptstyle m}\big\downarrow \\ \square^n \end{array} \quad \overset{u}{\longmapsto} \quad \begin{array}{c} A \times \square^1 \cup \square^n \\ \big\downarrow {\scriptstyle m \,\widehat{\times}\, \delta_k} \\ \square^n \times \square^1 \end{array}$$

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⊗ Generated by a set?

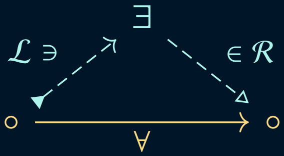
- ⊘ not constructively!
- ⊘ even classically, only for some choices of Φ

Small object arguments

⊗ Often don't start from a WFS, but from generators S:

$$\begin{aligned} \mathcal{R} &\coloneqq \{f \in \mathcal{E}^{\to} \mid f \text{ right lifts against all } m \in S\} \\ \mathcal{L} &\coloneqq \{m \in \mathcal{E}^{\to} \mid m \text{ left lifts against all } f \in \mathcal{R}\} \end{aligned}$$

⊗ But still need factorizations:



⊗ When S is "small" enough and $\mathcal{E}$ is well-behaved enough, can use small object argument(s) to build factorizations

Quillen's small object argument

- ⊗ Build a WFS from a set $\mathcal{S}$ of generators (assuming …)

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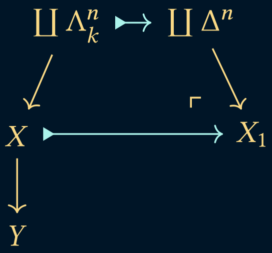

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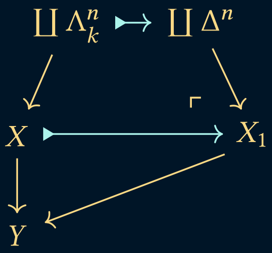

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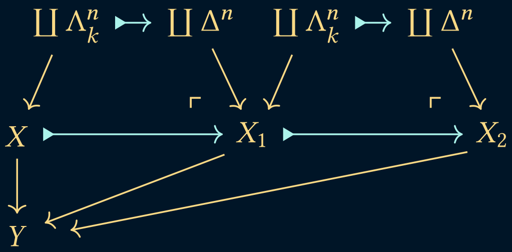

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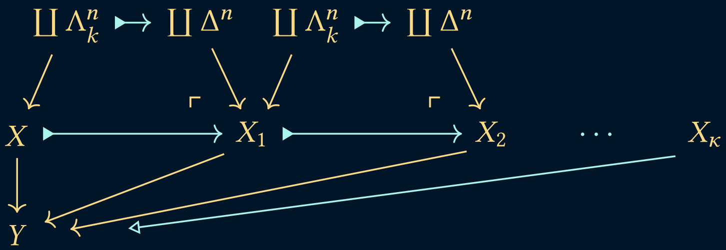

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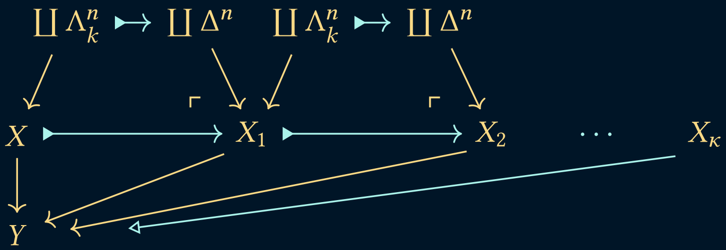

⊗ Left factor is a cell complex: a transfinite composite of cobase changes of coproducts of generators

⊗ By the "retract argument", any left map is a retract (in $\mathscr{E}^\to$) of one of these

Saturation

Def. $\mathcal{A} \subseteq \mathrm{Ob}\,\mathcal{E}^{\to}$ is saturated when it is closed under coproducts, cobase change, transfinite composition, and retracts.

- ⊗ The left class of any wfs is saturated.

- ⊗ When $(\mathcal{L}, \mathcal{R})$ is generated from $\mathcal{S}$ by the SOA, $\mathcal{L}$ is the least saturated class containing $\mathcal{S}$.

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- The left class of any wfs is saturated.
- When $(\mathcal{L}, \mathcal{R})$ is generated from $\mathcal{S}$ by the SOA, $\mathcal{L}$ is the least saturated class containing $\mathcal{S}$.
- Example usage: if $F\colon \mathcal{E} \to \mathcal{E}$ preserves coproducts, cobase changes and transfinite compositions and $F(S) \subseteq \mathcal{L}$, then $F(\mathcal{L}) \subseteq \mathcal{L}$.
- Our goal: find an equivalent for generation by a category!

Garner's algebraic small object argument

- Generates a weak factorization from a diagram $u\colon \mathscr{J} \to \mathscr{E}^{\to}$
- Advantages over Quillen's argument:
  - Factors through a standard free monad construction
  - Produces an algebraic weak factorization system

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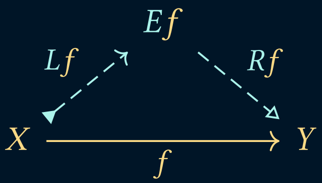

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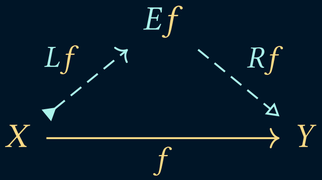
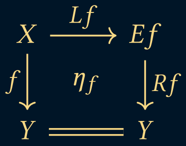


- R is a monad on $\mathscr{E}^{\to}$ with unit



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- Dually for L

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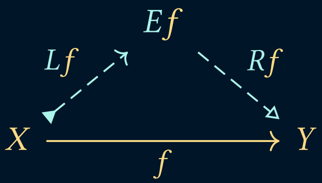

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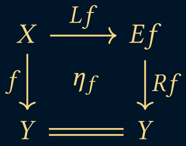


structure!

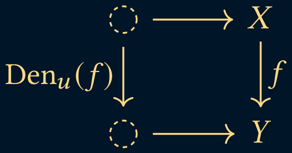
- the right maps are the (R, η)-algebras
- Dually for L

How does it work?: 1-step

- Use density comonad $\mathrm{Den}_u : \mathscr{E}^{\rightarrow} \rightarrow \mathscr{E}^{\rightarrow}$ for $u : \mathscr{J} \rightarrow \mathscr{E}^{\rightarrow}$

$$\mathrm{Den}_u(f) \coloneqq \mathrm{colim}_{j \in \mathscr{J}, \alpha : u_i \rightarrow f} \, u_i$$

“amalgamate all lifting problems against f”



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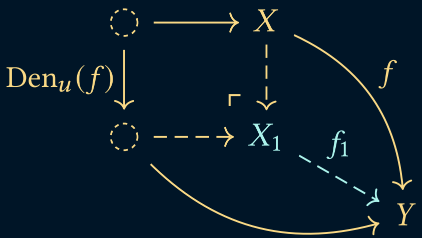

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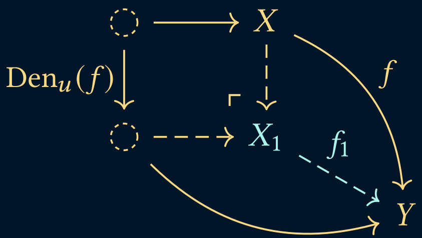

- ⊗ $R_1(f) \coloneqq f_1$ defines a pointed endofunctor on $\mathscr{E}^{\to}$
- ⊗ Rmk: when $\mathscr{J}$ is a set, $\mathrm{Den}_u(f)$ is a coproduct of generators

How does it work?: free monad

- Now take the free monad on the pointed endofunctor R_1
- How to do it?: A UNIFIED TREATMENT OF TRANSFINITE CONSTRUCTIONS FOR FREE ALGEBRAS, FREE MONOIDS, COLIMITS, ASSOCIATED SHEAVES, AND SO ON

  G.M. KELLY
- Given pointed endofunctor (T, τ) on $\mathscr{C}$, build free monad using

$$\circ \longrightarrow \circ \longrightarrow \cdots \longrightarrow \mathrm{colim}_{\alpha<\kappa}$$

- Idea in algebraic SOA case:
  repeatedly add solutions, but quotient out duplications

How does it work?: summary

generators $u\colon \mathscr{J} \to \mathscr{E}^{\to}$

$\Downarrow$

density comonad $\mathrm{Den}_u\colon \mathscr{E}^{\to} \to \mathscr{E}^{\to}$

$\Downarrow$

one-step factorization $R_1\colon \mathscr{E}^{\to} \to \mathscr{E}^{\to}$

$\Downarrow$

monad $R\colon \mathscr{E}^{\to} \to \mathscr{E}^{\to}$

$$\begin{array}{ccc} X & \xrightarrow{Lf} & Ef \\ {\scriptstyle f}\downarrow & \eta_f & \downarrow{\scriptstyle Rf} \\ Y & = & Y \end{array}$$

Is it a “saturation”?

- Builds Lf as a transfinite composite

$$X \longrightarrow X_1 \longrightarrow X_2 \longrightarrow \cdots \longrightarrow X_\kappa$$

  ... but step maps may not be left maps!

- Problem: collapsing duplicated solutions

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⊗ Problem: collapsing duplicated solutions

⊗ Important: can constrain these to a class $\mathcal{M}$ but need

- ⊘ ...
- ⊘ $\widehat{\mathrm{Den}_u} : (\mathscr{E}^{\to})^{\to} \to \mathscr{E}^{\to}$ sends levelwise $\mathcal{M}$ to $\mathcal{M}$

which often holds for monos but not left maps, "cofibrations"

Is it a "saturation"?

- ⊗ : don't look at maps $X_\alpha \to X_{\alpha+1}$, but $X \to X_{\alpha+1}$
- ⊗ See

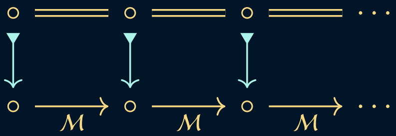
$$\begin{array}{ccccccccc} X & = & X & = & X & = & \cdots & = & X \\ \downarrow & & \downarrow & & \downarrow & & & & \downarrow \\ X & \longrightarrow & X_1 & \longrightarrow & X_2 & \longrightarrow & \cdots & \longrightarrow & X_\kappa \end{array}$$

as a colimit in the category of (L, ϵ)-coalgebras
(i.e., left-structured maps)

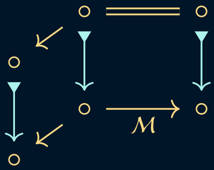
Cell complexes

- Lf is built using colimits in (L, ϵ)-Coalg:

(a) sequential colimits



(b) pushouts



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  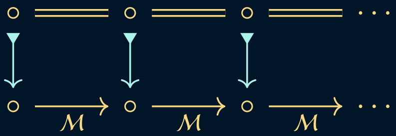

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  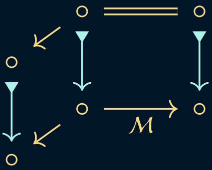

- and "vertical" composition $\circ \rightarrowtail \circ \rightarrowtail \circ$

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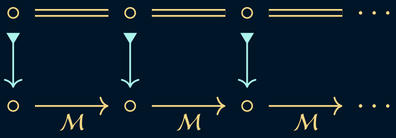

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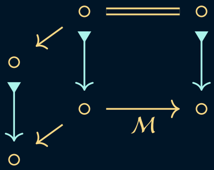
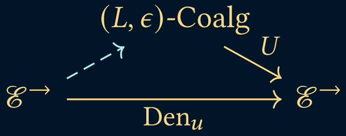


- and “vertical” composition 
- starting from coalgebras coming from the density comonad:



Universal property?

- Want the (L, ϵ)-coalgebras to be "smallest" with these structures
- Need composition of (L, ϵ)-coalgebras, so use Bourke–Garner double-categorical perspective

Universal property?

- ⊗ Want the (L, ϵ)-coalgebras to be "smallest" with these structures
- ⊗ Need composition of (L, ϵ)-coalgebras, so use Bourke–Garner double-categorical perspective
- ⊗ Have a double category (L, ϵ)-$\mathbb{C}$oalg where
  - ⊘ objects are objects $A \in \mathscr{E}$,
  - ⊘ horizontal morphisms are morphisms $f \colon A \longrightarrow B$ in $\mathscr{E}$,
  - ⊘ vertical morphisms are (L, ϵ)-coalgebras $m \colon A \mathrel{\rightarrow\!\!\!\!\bullet\,\,} B$,
  - ⊘ squares are morphisms of (L, ϵ)-coalgebras:

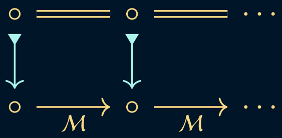
$$\begin{array}{ccc} A & \longrightarrow & A' \\ {\scriptstyle m}\big\downarrow\!\bullet & & \bullet\!\big\downarrow{\scriptstyle m'} \\ B & \longrightarrow & B' \end{array}$$

- ⊗ Call something like this a vertical structure on $\mathscr{E}$

Universal property

Thm: Fix $u\colon \mathscr{J} \to \mathscr{E}^{\to}$ for which algebraic SOA applies
and "suitable" $\mathcal{M} \subseteq \mathrm{Ob}\,\mathscr{E}^{\to}$.
Suppose $\mathbb{A}$ is a vertical structure on $\mathscr{E}$ with

(a) sequential colimits



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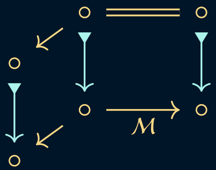
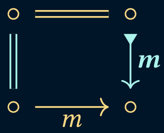
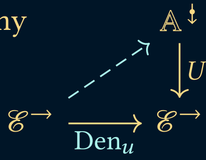


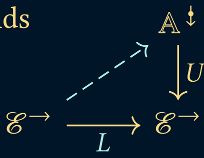
(c) "left connections"



Then any



yields



(+ structure that makes it essentially unique)

Is it useful?

- For uniform fibrations, $\mathrm{Den}_u : \mathscr{E}^{\to} \to \mathscr{E}^{\to}$ sends

$$\begin{array}{c} X \\ {\scriptstyle f}\downarrow \\ Y \end{array} \quad \overset{\mathrm{Den}_u}{\longmapsto} \quad \begin{array}{c} A \times \square^1 \cup B \\ \downarrow {\scriptstyle m_0 \,\hat{\times}\, \delta_0} \\ B \times \square^1 \end{array} \;\sqcup\; \begin{array}{c} A \times \square^1 \cup B \\ \downarrow {\scriptstyle m_1 \,\hat{\times}\, \delta_1} \\ B \times \square^1 \end{array}$$

This is the kind of reduction we need!

- Our example: for $F : \mathscr{E} \to \mathscr{E}$ take $\mathbb{A} = F^{-1}((L, \epsilon)\text{-}\mathbb{C}\mathrm{oalg})$

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$$\begin{array}{c} X \\ f \big\downarrow \\ Y \end{array} \quad \overset{\mathrm{Den}_u}{\longmapsto} \quad \begin{array}{c} A \times \square^1 \cup B \\ \big\downarrow m_0 \,\widehat{\times}\, \delta_0 \\ B \times \square^1 \end{array} \;\sqcup\; \begin{array}{c} A \times \square^1 \cup B \\ \big\downarrow m_1 \,\widehat{\times}\, \delta_1 \\ B \times \square^1 \end{array}$$

This is the kind of reduction we need!

- Our example: for $F : \mathscr{E} \to \mathscr{E}$ take $\mathbb{A} = F^{-1}((L, \epsilon)\text{-}\mathbb{C}\mathrm{oalg})$

- For (complemented mono, split epi)
or (cofibration, trivial fibration),
usually all left maps are in image of Den_u — useless!

Last remarks

⊗ Another application: building extension operations:

$$
\begin{array}{ccc}
X & \dashrightarrow & \exists X' \\
\forall \downarrow \ \lrcorner & & \downarrow \\
C & \rightarrowtail & D
\end{array}
\quad \Longleftarrow \quad
\begin{array}{ccc}
X & \dashrightarrow & \exists X' \\
\forall \downarrow \ \lrcorner & & \downarrow \\
A \times \square^1 \cup B & \rightarrowtail & B \times \square^1
\end{array}
$$

⊘ key property for building cubical (and other) model structures

⊘ usually use a universe for this reduction
—but do we need to? (no)

Last remarks

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\end{array}
$$

  - key property for building cubical (and other) model structures
  - usually use a universe for this reduction
    —but do we need to? (no)

- I forgot about retracts
  - With sufficiently structured retract lifting in $\mathbb{A}$, get
    (L, ϵ)-Coalg $\longrightarrow \mathbb{A}^{\downarrow}$ or even (L, ϵ)-$\mathbb{C}$oalg $\longrightarrow \mathbb{A}$
  - ... but don't know if these are "universal" (talk to me!)

The End!