Geometric homotopy theory via simplicial sets

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..., or: How not to do synthetic homotopy theory

> *Univalent foundations has a very different "synthetic" approach to homotopy theory in which homotopy types are the basic objects (namely, the types) rather than being constructed using topological spaces or some other set-theoretic model.* — The HoTT Book (2013)

Can also do homotopy theory the "traditional way" inside HoTT, using topological spaces, simplicial sets, etc., if it can be made constructive. This might be useful to access classical objects for which we don't yet know of a synthetic construction.

For example: cohomology theories that arise from geometry, like topological K-theory and complex cobordism.

Why bother? / Why not use e.g. real-cohesion?

Theorem (Adams–Atiyah, ’66)

If $\mathbb{S}^{n-1}$ has an H-space structure, then $n = 1, 2, 4,$ or 8.

One-page proof using topological K-theory (and the Adams operations).

The theorem statement just involves spheres, and is easily expressed in synthetic homotopy theory. So, we’d ideally be able to replicate the *proof* in book HoTT, rather than in an extension designed to talk about geometry.

If we can’t think of any synthetic approach, we might instead try to:

1. build a simplicial set model for the Bott map $\mathbb{Z} \times BU \xrightarrow{\sim} \Omega^2(\mathbb{Z} \times BU)$;
2. take the homotopy type of this map to obtain topological K-theory.

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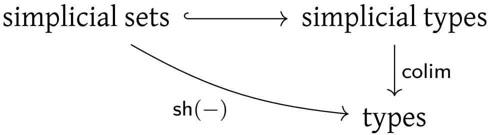
The shape problem

We'd need an internal construction that sends, e.g., a simplicial set like the simplex boundary $\partial\Delta^n$ to its shape, the "synthetic" sphere $\mathbb{S}^{n-1} : \mathsf{Type}$.

We don't know whether this is possible in book (or cubical) HoTT: problem of coherence data in every dimension. However, in some extensions of HoTT this is possible: e.g., ones with (semi)simplicial types or ∞-categories.



Let's assume for the moment that we have some construction sending a simplicial set X to its shape $\mathsf{sh}(X)$.

The control problem

The plan was to build a map of simplicial sets

$$\Phi : \mathbb{Z} \times BU \to \Omega^2(\mathbb{Z} \times BU)$$

and take its shape to get K-theory. Here, Ω denotes a simplicial model for the loop space. But the shape of Φ is a map (of types)

$$\mathsf{sh}(\Phi) : \mathsf{sh}(\mathbb{Z} \times BU) \to \mathsf{sh}(\Omega^2(\mathbb{Z} \times BU))$$

whereas to build the K-theory spectrum, we want a map

$$? : \mathsf{sh}(\mathbb{Z} \times BU) \to \Omega^2(\mathsf{sh}(\mathbb{Z} \times BU))$$

Question

Does $\Omega \circ \mathsf{sh} = \mathsf{sh} \circ \Omega$? More broadly, to what extent does the shape preserve pushout/pullback squares of simplicial sets?

Overview of this talk

This talk will propose a few axioms on a subuniverse $\mathcal{S}$ of "shapes" together with an "interval" type $\mathbb{I}$, and use them to address these questions.

1. Overall plan of attack
2. The axioms
3. Simplicial homotopy theory
4. Geometric input

Actually constructing topological K-theory is still future work.

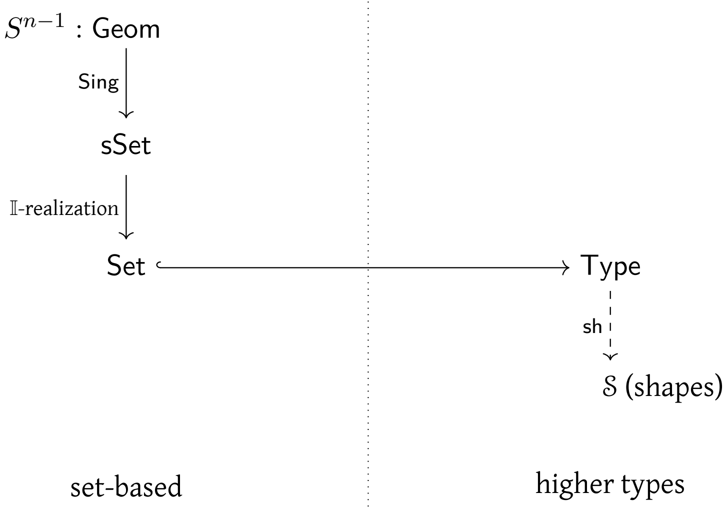
General approach

The whole development will take place in ordinary HoTT augmented by a few axioms. As usual, by "set" we mean a type that is an h-set.

We make free use of any set-based mathematics that is constructively valid: notably, the simplex category Δ, the (univalent) category $\mathsf{sSet} := \mathsf{Set}^{\Delta^{\mathrm{op}}}$ of simplicial sets, and the *constructive Kan–Quillen model structure* on sSet [Henry '19; Gambino–Sattler–Szumiło '22]. (Later, we'll also need a constructive description of geometric objects like unitary groups.)

By building this model category inside HoTT, we gain the ability to express the relationship between a simplicial set and its shape: the homotopy type it represents.

The pipeline



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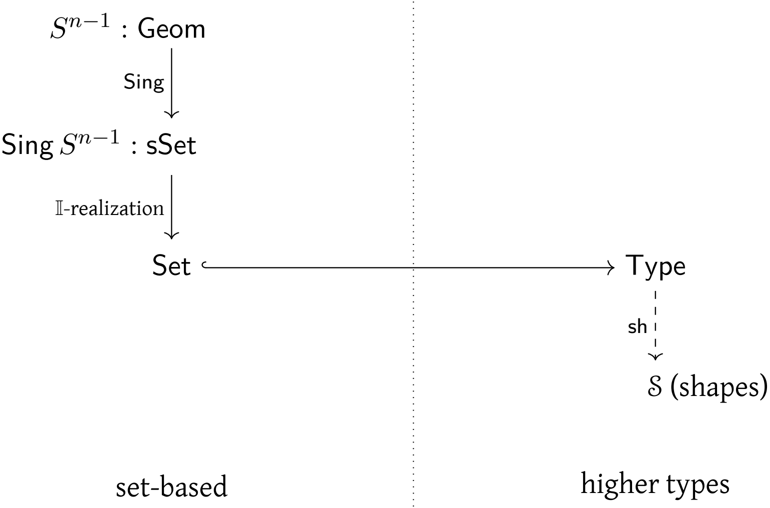

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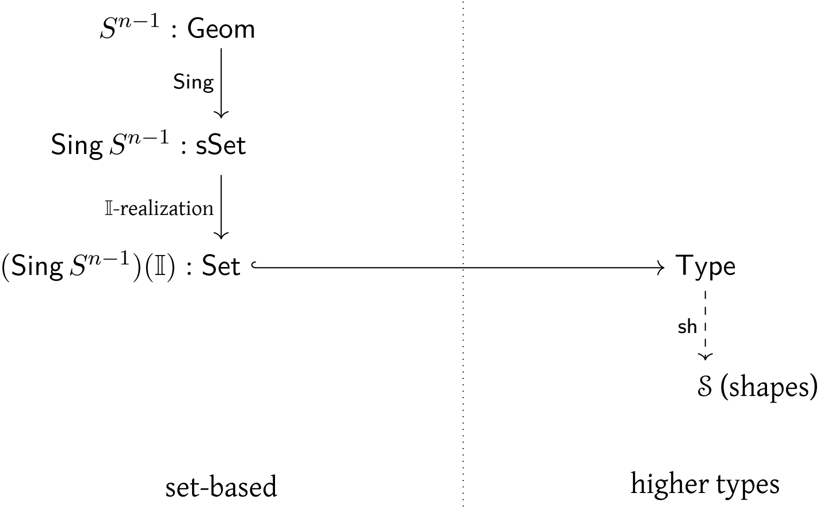


The pipeline

S^{n-1} : Geom

$\downarrow$ Sing

$\mathsf{Sing}\, S^{n-1}$: sSet

$\downarrow$ $\mathbb{I}$-realization

$(\mathsf{Sing}\, S^{n-1})(\mathbb{I})$: Set $\hookrightarrow$ Type

Type $\dashrightarrow$ (sh)

$\mathsf{sh}(\cdots) = \mathbb{S}^{n-1} : \mathcal{S}$ (shapes)

set-based | higher types

Axioms I: Shapes

Axioms on shapes:

1. There is a subuniverse $\mathcal{S} \subseteq \mathsf{Type}$ of shapes, closed under Σ-types, finite limits, and finite colimits. We also call shapes discrete (especially sets).

Then $\mathcal{S}$ also contains the types 0, 1, 2, $\mathbb{S}^1$ (and the rest of the spheres), and $\Omega\mathbb{S}^1 = \mathbb{N}$; and therefore is closed under sequential colimits, since these can be built using $\mathbb{N}$-indexed Σ-types and coequalizers.

"Shape" is intended in the sense of cohesion. However, we do *not* assume that $\mathcal{S}$ is a reflective subuniverse (let alone associated to any modality).

A remark: The propositions in $\mathcal{S}$ form a *dominance* $(\Sigma, \top)$ closed under countable disjunctions, like the open propositions in synthetic topology.

The shape of a type

We call a function $f : X \to S$ with $X : \mathsf{Type}$ and $S : \mathcal{S}$ a shape map if it is a reflection of X into $\mathcal{S}$. Explicitly, this means for each shape $T : \mathcal{S}$, the map

$$f^* : (S \to T) \to (X \to T)$$

is an equivalence.

We don't assume $\mathcal{S}$ is a reflective subuniverse, so a type X might not admit a shape map, but if it does, we say that X "has a shape" and denote its (essentially unique) shape map by $\eta_X : X \to \mathsf{sh}(X)$.

$\mathsf{sh}(-)$ is the "partially defined reflection" into $\mathcal{S}$. Since $\mathcal{S}$ is closed under pushouts and sequential colimits, $\mathsf{sh}(-)$ preserves these colimits.

Example: Simplicial spaces

At the level of objects:

$$\begin{aligned} \text{types} &= \text{simplicial spaces} \\ \text{shapes} &= \text{constant simplicial spaces, i.e., } \infty\text{-groupoids} \\ \mathsf{sh}(X) &= \text{groupoid reflection (= geometric realization) of } X \end{aligned}$$

Optionally, we could restrict the shapes to *countable* CW complexes, which are closed under Σ-types and finite (co)limits. In this case, a type might fail to have a shape because it is too big.

Example: Condensed anima

At the level of objects:

$$\begin{aligned} \text{types} &= \text{condensed anima } (\infty\text{-groupoids}) \\ \text{sets} &= \text{condensed sets, such as compact Hausdorff spaces} \\ \text{shapes} &= \text{constant condensed anima (optionally assumed countable)} \\ \text{discrete sets} &= \text{actual sets (optionally assumed countable)} \end{aligned}$$

The shape of the topological circle S^1 is the ∞-groupoid $\mathbb{S}^1$.

This example is definitely *not* cohesive: a non-locally contractible space like the Cantor set will not have a reflection into the subcategory of shapes.

This is the motivating example for the whole project, inspired by the proof of the Brouwer fixed-point theorem in Synthetic Stone Duality [CCGM24].

Axioms II: The interval type

Axioms on the interval:

2. There is a set $\mathbb{I}$ equipped with two points $0,\ 1 : \mathbb{I}$, such that $\neg(0 =_{\mathbb{I}} 1)$.

2′. The set $\mathbb{I}$ is further equipped with a linear ordering $\leq : \mathbb{I} \to \mathbb{I} \to \mathsf{Prop}$, with least element 0 and greatest element 1.

3. The set $\mathbb{I}$ is shape-contractible, meaning $\mathbb{I} \to 1$ is a shape map (so $\mathsf{sh}(\mathbb{I}) = 1$).

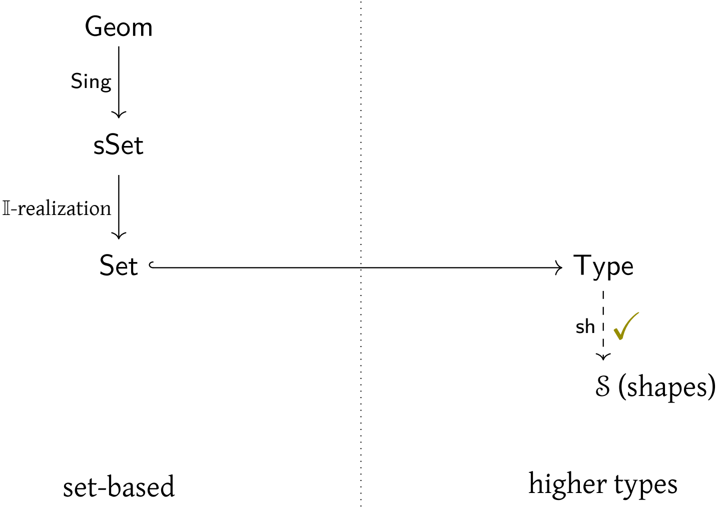
Examples: $\mathbb{I} = \Delta^1$ in simplicial spaces; $\mathbb{I} = [0, 1] \subseteq \mathbb{R}$ in condensed anima.

Conjecture

We can get rid of axiom 2′ (maybe replacing it by a connection).

Then we'd have to do everything with cubical sets instead of simplicial sets.

The pipeline



$\mathbb{I}$-realization

Theorem (Joyal?, Mac Lane–Moerdijk)

There is a unique functor $-(\mathbb{I}) : \mathsf{sSet} \to \mathsf{Set}$ *preserving colimits and sending* Δ^n *to*

$$\Delta^n(\mathbb{I}) := \{ (x_1, \ldots, x_n) \mid x_1 \leq \cdots \leq x_n \} \subseteq \mathbb{I}^n.$$

Furthermore, $-(\mathbb{I})$ *preserves finite limits. We call* $K(\mathbb{I}) : \mathsf{Set}$ *the* $\mathbb{I}$*-realization of* K*.*

In condensed anima ($\mathbb{I} := [0, 1] \subseteq \mathbb{R}$), if $K : \mathsf{sSet}$ is levelwise discrete (each K_n is an ordinary set), then $K(\mathbb{I})$ is the ordinary geometric realization of K.

Definition

We define the *shape* of $K : \mathsf{sSet}$ to be $\mathsf{sh}(K(\mathbb{I})) : \mathcal{S}$ (provided this exists).

Lemma

$\mathsf{sh}(\Delta^n) = 1$.

Cofibrancy

In the constructive Kan–Quillen model structure, K : sSet is (Reedy) *cofibrant* if for each $n : \mathbb{N}$, the degenerate n-simplices form a decidable subset of K_n, with complement $N_n K$.

In this case, K can be expressed as the sequential colimit of its skeleta

$$K = \mathsf{colim}\left(\varnothing = \mathsf{sk}^{-1}(K) \hookrightarrow \mathsf{sk}^{0}(K) \hookrightarrow \mathsf{sk}^{1}(K) \hookrightarrow \cdots\right)$$

and the skeleta are built inductively as pushouts of boundary inclusions:

$$\begin{array}{ccc} N_n K \cdot \partial\Delta^n & \longrightarrow & \mathsf{sk}^{n-1} K \\ \downarrow & & \downarrow \\ N_n K \cdot \Delta^n & \longrightarrow & \mathsf{sk}^{n} K \end{array}$$

We call this the *cellular presentation* of K.

Shape of a simplicial set

Proposition

If K : sSet *is cofibrant and levelwise discrete then* $\mathsf{sh}(K)$ *exists.*

Proof.

Induction on the cellular presentation of K, using the fact that shape preserves sequential colimits and pushouts *computed in types*. By definition,

$$\begin{array}{ccc} N_n K \cdot \partial\Delta^n(\mathbb{I}) & \longrightarrow & (\mathsf{sk}^{n-1} K)(\mathbb{I}) \\ \downarrow & \ulcorner & \downarrow \\ N_n K \cdot \Delta^n(\mathbb{I}) & \longrightarrow & (\mathsf{sk}^{n} K)(\mathbb{I}) \end{array}$$

is a pushout square *of sets*. But $\partial\Delta^n \hookrightarrow \Delta^n$ is a mono, and $\mathbb{I}$-realization is left exact, so the vertical maps are monos and the square is also a pushout in Type. $N_n K$ is discrete because it is decidable subtype of $K_n : \mathbb{S}$. □

Kan fibrations

In the constructive Kan–Quillen model structure, a *Kan fibration* $p : K \to L$ is a map with chosen solutions to all lifting problems against horn inclusions.

Proposition

If

$$\begin{array}{ccc} K' & \longrightarrow & K \\ {\scriptstyle p'}\downarrow & \lrcorner & \downarrow{\scriptstyle p} \\ L' & \longrightarrow & L \end{array}$$

is a pullback square with p *(hence also* p'*) Kan fibrations and all objects cofibrant and levelwise discrete, then* sh *sends this square to a pullback in* $\mathcal{S}$*.*

In brief: sh sends homotopy pullbacks to pullbacks of shapes.

Kan fibrations: proof sketch

This is the main technical difficulty:

- define a class of maps called *quasifibrations* (as in [Dold–Thom '58]);
- reduce to showing the realization of a Kan fibration is a quasifibration;
- prove gluing lemmas for quasifibrations, using descent properties of pushouts and sequential colimits;
- thereby reduce to the case of a Kan fibration $p : K \to \Delta^n$;
- use model category arguments to prove that such a fibration is fiberwise homotopy equivalent to a "constant" one $K' \times \Delta^n \to \Delta^n$;
- deduce that every fiber of $p(\mathbb{I}) : K(\mathbb{I}) \to \Delta^n(\mathbb{I})$ is $\mathbb{I}$-homotopy equivalent to $K(\mathbb{I})$ itself, making $p(\mathbb{I})$ a quasifibration.

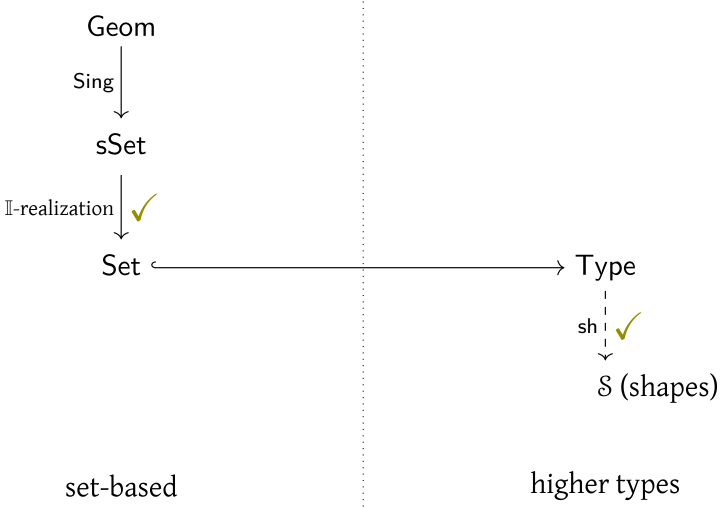
Summary of simplicial homotopy theory

- If K : sSet is cofibrant (degeneracy is decidable) and levelwise discrete (each K_n is in $\mathcal{S}$) then we can form its shape $\mathsf{sh}(K) := \mathsf{sh}(K(\mathbb{I})) : \mathcal{S}$.
- Shape sends pullbacks of Kan fibrations to pullbacks in $\mathcal{S}$.

Using these facts, we can show that if K is cofibrant, levelwise discrete and a Kan complex, then

$$\mathsf{sh}(\Omega K) = \Omega\,\mathsf{sh}(K).$$

The pipeline



Geometric input

How to turn a "geometric object" like the $(n-1)$-sphere

$$S^{n-1} := \{ (x_1, \ldots, x_n) \mid x_1^2 + \cdots + x_n^2 = 1 \}$$

into such a cofibrant, levelwise discrete Kan complex?

Traditional homotopy answer is to use $\mathsf{Sing} : \mathsf{Top} \to \mathsf{sSet}$:

$$\begin{aligned}(\mathsf{Sing}\, X)_k &:= \mathsf{Top}(|\Delta^k|, X), \\ |\Delta^k| &:= \{ (t_1, \ldots, t_k) \in \mathbb{R}^k \mid 0 \le t_1 \le \cdots \le t_k \le 1 \} \in \mathsf{Top}.\end{aligned}$$

That's unlikely to work here because degeneracy will not be decidable constructively, and the real numbers may not be discrete.

Semialgebraic maps

Rather than using arbitrary continuous functions, we use only the semialgebraic ones, those whose graphs are defined by first-order formulas in the language of a ring, without parameters from $\mathbb{R}$.

$$(\mathsf{Sing}\, S^{n-1})_k := \big\{ \text{formulas } \phi(\vec{t}, \vec{x}) \text{ in } k+n \text{ variables defining the graph of a continuous function } |\Delta^k| \to S^{n-1} \big\}/{\sim}$$

where

$$\big(\phi(\vec{t}, \vec{x}) \sim \psi(\vec{t}, \vec{x})\big) := \big(\mathbb{R} \vDash \forall\, \vec{t}\, \vec{x}.\, \phi(\vec{t}, \vec{x}) \Leftrightarrow \psi(\vec{t}, \vec{x})\big).$$

This is a cofibrant, levelwise discrete simplicial set by Tarski's theorem: the first-order theory of $\mathbb{R}$ (as a ring) is decidable. It has the same homotopy type as the usual Sing S^{n-1} by a theorem of [Delfs–Knebusch '85].

Conclusions

- Synthetic homotopy theory is really easier, when we can do it!
- But if not, then we can also do traditional homotopy theory, and connect it to the synthetic theory in a mild extension of HoTT
- Building model categories internal to HoTT lets us express the relationship between objects of the model category and the types they present, which may clarify constructive set-based homotopy theory.

Thank you!