The Cantor–Schröder–Bernstein theorem in ∞-Topoi

Fredrik Bakke
Norwegian University of Science and Technology

April 15, 2025

Introduction

Theorem (Cantor–Schröder–Bernstein)

If two sets mutually inject they are in bijection.

- Simple statement
- Fundamental result
- But classical

History/Context

1887 Stated by Cantor without proof.

1887 Proven by Dedekind assuming only LEM. (unpublished)

1895 Proven by Cantor assuming the well-ordering theorem.

1896 Proven by Bernstein assuming only LEM. (published)

⋮

2019 Pradic and Brown show LEM follows from theorem. [PB22]

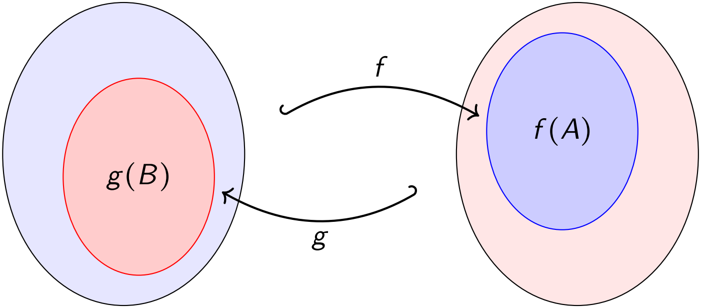
2020 Escardó generalizes theorem to boolean ∞-toposes. [Esc21]

2023 Forster–Jahn–Smolka give an entirely axiomless construction for retracts of $\mathbb{N}$. [FJS23]

The Cantor–Schröder–Bernstein theorem

Theorem

If two sets A and B mutually inject, they are in bijection.



Proof sketch (König)

Given injections $f : A \hookrightarrow B$ and $g : B \hookrightarrow A$.

1. For any $x : A$, we can by the law of excluded middle decide if x has a (necessarily unique) preimage under g, and if so decide if f has a preimage of that, and so on.
2. Can form the potentially-infinite chain

$$x,\ g^{-1}(x),\ f^{-1}(g^{-1}(x)),\ g^{-1}(f^{-1}(g^{-1}(x))),\ \ldots$$

   We say x is a *perfect image* of g relative to f if it is the case that, for every element in A in this chain we can always produce a preimage under g.
3. We define a new map $h : A \to B$ by

$$h(x) :\equiv g^{-1}(x) \text{ if } x \text{ is perfect, otherwise } h(x) :\equiv f(x).$$

   We can argue by case analysis, using the properties of perfect images, that this map is an equivalence. □

Proof sketch (König)

Given injections $f : A \hookrightarrow B$ and $g : B \hookrightarrow A$.

1. For any $x : A$, we can by the law of excluded middle decide if x has a (necessarily unique) preimage under g, and if so decide if f has a preimage of that, and so on.
2. Can form the potentially-infinite chain

$$x,\ g^{-1}(x),\ f^{-1}(g^{-1}(x)),\ g^{-1}(f^{-1}(g^{-1}(x))),\ \ldots$$

   We say x is a *perfect image* of g relative to f if it is the case that, for every element in A in this chain we can always produce a preimage under g.

3. We define a new map $h : A \to B$ by

$$h(x) :\equiv g^{-1}(x) \text{ if } x \text{ is perfect, otherwise } h(x) :\equiv f(x).$$

   We can argue by case analysis, using the properties of perfect images, that this map is an equivalence. □

Proof sketch (König)

Given injections $f : A \hookrightarrow B$ and $g : B \hookrightarrow A$.

1. For any $x : A$, we can by the law of excluded middle decide if x has a (necessarily unique) preimage under g, and if so decide if f has a preimage of that, and so on.
2. Can form the potentially-infinite chain

   $$x,\ g^{-1}(x),\ f^{-1}(g^{-1}(x)),\ g^{-1}(f^{-1}(g^{-1}(x))),\ \ldots$$

   We say x is a *perfect image* of g relative to f if it is the case that, for every element in A in this chain we can always produce a preimage under g.
3. We define a new map $h : A \to B$ by

   $$h(x) :\equiv g^{-1}(x) \text{ if } x \text{ is perfect, otherwise } h(x) :\equiv f(x).$$

   We can argue by case analysis, using the properties of perfect images, that this map is an equivalence. □

Analysis of assumptions

1. "Preimages are unique since A and B are h-sets."
   $\rightsquigarrow$ assume f and g are embeddings [Esc21]

2. "Decide if f and g have preimages."
   $\rightsquigarrow$ add condition locally

3. "Decide if an element is a perfect image."
   $\rightsquigarrow$ WLPO

Analysis of assumptions

1. "Preimages are unique since A and B are h-sets."
   $\rightsquigarrow$ assume f and g are embeddings [Esc21]

2. "Decide if f and g have preimages."
   $\rightsquigarrow$ add condition locally

3. "Decide if an element is a perfect image."
   $\rightsquigarrow$ WLPO

Analysis of assumptions

1. "Preimages are unique since A and B are h-sets."
   $\rightsquigarrow$ assume f and g are embeddings [Esc21]

2. "Decide if f and g have preimages."
   $\rightsquigarrow$ add condition locally

3. "Decide if an element is a perfect image."
   $\rightsquigarrow$ WLPO

The weak limited principle of omniscience

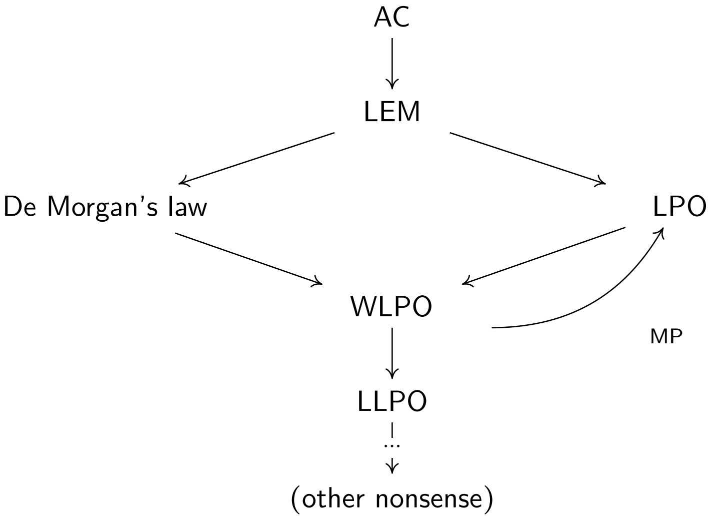
Definition

The **weak limited principle of omniscience** asserts any of

1. Given a decidable subtype of $\mathbb{N}$, it is decidable if it is full.
2. Given any binary sequence $\mathbb{N} \to \{0, 1\}$, it is decidable if it is constant.
3. It is decidable if an element of the conatural numbers $\mathbb{N}_\infty$ is infinite.
4. Given a family of decidable types P over $\mathbb{N}$, the type of sections $(n : \mathbb{N}) \to P\, n$ is (proof-relevantly) decidable.

This is an anti-topological principle!

Some constructive taboos



Theorem without LEM

Theorem

Assuming WLPO, if g and f are decidable embeddings then $A \simeq B$.

In fact, already if the fibers of f are decidable and have double negation dense equality

$$(p\,q : \mathrm{fiber}\,f\,x) \to \neg\neg(p = q),$$

then B is a retract of A.

Well, actually, if we can already decide if any element is a perfect image, then g needs only be a double negation stable embedding, and the fibers of f need only satisfy the property that total spaces of double negation stable subtypes satisfy double negation elimination

$$(P : \mathrm{fiber}\,f\,x \to \Omega_{\neg\neg}) \to \neg\neg\Sigma P \to \Sigma P.$$

Theorem without LEM

Theorem

Assuming WLPO, if g and f are decidable embeddings then $A \simeq B$.

In fact, already if the fibers of f are decidable and have double negation dense equality

$$(p\, q : \mathrm{fiber}\, f\, x) \to \neg\neg(p = q),$$

then B is a retract of A.

Well, actually, if we can already decide if any element is a perfect image, then g needs only be a double negation stable embedding, and the fibers of f need only satisfy the property that total spaces of double negation stable subtypes satisfy double negation elimination

$$(P : \mathrm{fiber}\, f\, x \to \Omega_{\neg\neg}) \to \neg\neg\Sigma P \to \Sigma P.$$

Theorem without LEM

Theorem

Assuming WLPO, if g and f are decidable embeddings then $A \simeq B$.

In fact, already if the fibers of f are decidable and have double negation dense equality

$$(p\, q : \mathrm{fiber}\, f\, x) \to \neg\neg(p = q),$$

then B is a retract of A.

Well, actually, if we can already decide if any element is a perfect image, then g needs only be a double negation stable embedding, and the fibers of f need only satisfy the property that total spaces of double negation stable subtypes satisfy double negation elimination

$$(P : \mathrm{fiber}\, f\, x \to \Omega_{\neg\neg}) \to \neg\neg\Sigma P \to \Sigma P.$$

Theorem for “finite” types

Theorem

For most notions of finiteness, if A and B are finite types that mutually embed, we have $A \simeq B$.*

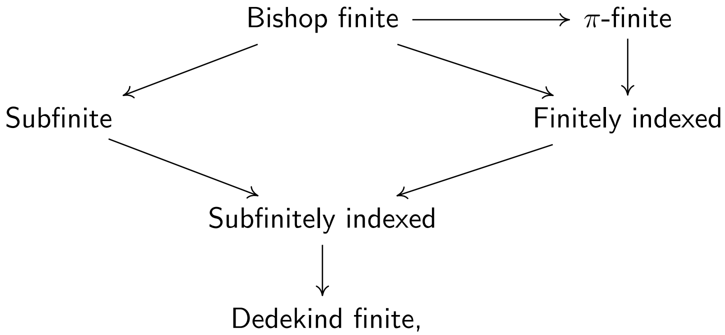
*applies to all of

Theorem for "finite" types

Theorem

For most notions of finiteness, if A and B are finite types that mutually embed, we have $A \simeq B$.*

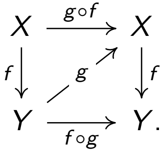
*applies to all of



Theorem for "finite" types — proof

Recall that a type X is **Dedekind finite** if every self-embedding $X \hookrightarrow X$ is an equivalence.

Now, assume that we are given a pair of mutually embedding Dedekind finite types $f : X \hookrightarrow Y$ and $g : Y \hookrightarrow X$, then we have a commuting diagram



By Dedekind finiteness the top and bottom rows are equivalences, so by the 6-for-2 property f and g are equivalences. □

Further questions

1. Can we give nondegenerate examples of proper retracts with the construction?
2. Can we prove an entirely axiomless version with no assumptions on A or B?
3. Forster–Jahn–Smolka give an axiomless construction for retracts of $\mathbb{N}$. Can we extend this approach to other domains?
4. Dual: when can we conclude that $A \simeq B$ given *epimorphisms* $A \twoheadrightarrow B$ and $B \twoheadrightarrow A$?

Conclusion

Formalization: TypeTopology PR#351

- Cantor–Schröder–Bernstein is a fundamental, but classical theorem.
- Can be generalized to some extent.
- Can we do better?

Thank you!

References

[Esc21] Martín Hötzel Escardó. "The Cantor-Schröder-Bernstein theorem for ∞-groupoids". In: **J. Homotopy Relat. Struct.** 16.3 (2021), pp. 363–366. ISSN: 2193-8407,1512-2891. DOI: 10.1007/s40062-021-00284-6.

[FJS23] Yannick Forster, Felix Jahn, and Gert Smolka. "A Computational Cantor-Bernstein and Myhill's Isomorphism Theorem in Constructive Type Theory (Proof Pearl)". In: **Proceedings of the 12th ACM SIGPLAN International Conference on Certified Programs and Proofs**. CPP 2023. Association for Computing Machinery, 2023, pp. 159–166. ISBN: 9798400700262. DOI: 10.1145/3573105.3575690.

[PB22] Cécilia Pradic and Chad E. Brown. **Cantor-Bernstein implies Excluded Middle**. Aug. 2022. arXiv: 1904.09193 [math.LO].