A note on left adjoints preserving colimits in HoTT

Perry Hart Kuen-Bang Hou (Favonia)

University of Minnesota, Twin Cities

In classical category theory, a well-known fact is that *left adjoints preserve colimits (LAPC)*. We would like to port this theorem to adjunctions between wild categories in HoTT. However, this turns out to be harder than one might hope. We isolate a sufficient condition for the proof to go through. With this condition, combined with an extension of a known technique based on homogeneous types, we show that suspension, an example of a left adjoint, preserves colimits (over graphs). Note: the sufficient condition and application to the suspension functor were already mentioned in our prior work [4, Appendix B], but here we place these results in a wider context and explain, for the first time, how we formally prove that suspension preserves colimits.

Classical proofs of LAPC

Consider an adjunction $L \dashv R$ between 1-categories $\mathcal{C}$ and $\mathcal{D}$. Let $\mathcal{J}$ be a small 1-category. The classical theorem states that L preserves $\mathcal{J}$-shaped colimits, and it has two well-known proofs. The first requires that $\mathcal{C}$ and $\mathcal{D}$ admit global colimit functors $\mathrm{colim}_{\mathcal{J}} : \mathcal{C}^{\mathcal{J}} \to \mathcal{C}$ and $\mathrm{colim}_{\mathcal{J}} : \mathcal{D}^{\mathcal{J}} \to \mathcal{D}$ [7, Chapter V.5]. The proof assumes these two colimit functors satisfy some coherence conditions that are automatically true for 1-categories (but *not* for wild ones).

Instead of requiring global colimit functors, the second proof starts with a specific colimit $\mathrm{colim}_{\mathcal{J}}(F)$ of a diagram $F : \mathcal{J} \to \mathcal{C}$ and shows that $L(\mathrm{colim}_{\mathcal{J}}(F))$ is the expected colimit. Similarly to the first proof, it also assumes some coherence conditions that are true for 1-categories, and thus it is expected that we will need further assumptions when working with wild categories. Now, the proof argues that for each $Y \in \mathsf{Ob}(\mathcal{D})$, the following chain of isomorphisms with $C :\equiv \mathrm{colim}_{\mathcal{J}}(F)$ equals the canonical post-composition map [8, Theorem 4.5.2]:

$$\mathsf{hom}_{\mathcal{D}}(L(C), Y) \cong \mathsf{hom}_{\mathcal{C}}(C, R(Y)) \cong \mathsf{lim}_i(\mathsf{hom}_{\mathcal{C}}(F_i, R(Y))) \cong \mathsf{lim}_i(\mathsf{hom}_{\mathcal{D}}(L(F_i), Y)) \qquad \texttt{(iso)}$$

This means that the induced cocone on $L(\mathrm{colim}_{\mathcal{J}}(F))$ is indeed colimiting, i.e., L preserves colimits. Besides avoiding global colimit functors, this proof argues mostly in terms of hom-isomorphisms that can be directly supplied by a reasonable definition of adjunctions. This helps us formulate further conditions needed for wild categories. Thus, we will port the second proof to HoTT.

Porting to wild categories in HoTT

Let $\mathcal{C}$ and $\mathcal{D}$ be wild categories with functors $L : \mathcal{C} \to \mathcal{D}$ and $R : \mathcal{D} \to \mathcal{C}$. Let Γ be a graph and $F : \Gamma \to \mathcal{C}$ be a $\mathcal{C}$-valued diagram over Γ. Consider a cocone (C, r, K) under F where $r_i : F_i \to C$ for vertices $i : \Gamma_0$ and $K_{i,j,g} : r_j \circ F_{i,j,g} = r_i$ for $i, j : \Gamma_0$ and edges $g : \Gamma_1(i, j)$. As notation abuse, we write $L(K_{i,j,g})$ as the witness for $L(r_j) \circ L(F_{i,j,g}) = L(r_i)$ derived from the composition law $L_\circ$ of L.

Suppose this cocone (C, r, K) is *colimiting*. Further, suppose that $L \dashv R$, witnessed by a family of hom-equivalences $\alpha : \prod_{X:\mathsf{Ob}(\mathcal{D})} \prod_{A:\mathsf{Ob}(\mathcal{C})} \mathsf{hom}_{\mathcal{D}}(LA, X) \xrightarrow{\simeq} \mathsf{hom}_{\mathcal{C}}(A, RX)$ together with proofs V_1 and V_2 of the naturality of α in X and A, respectively. We wish to replay the second classical proof by showing the chain of isomorphisms `(iso)` equals the canonical post-composition map.

Let ζ denote the composite of these isomorphisms. By function extensionality, it suffices to show ζ and post-composition are equal on h for every $h : \mathsf{hom}_{\mathcal{D}}(L(C), Y)$. For each $i : \Gamma_0$, we can build an equality $Q_i : \alpha^{-1}(\alpha(h) \circ r_i) = h \circ L(r_i)$ from V_2 and the equivalence data for α. By the structure identity principle for lim, it suffices to show the following equality for all $i, j : \Gamma_0$ and $g : \Gamma_1(i, j)$:

$$\mathsf{ap}_{-\circ L(F_{i,j,g})}(Q_j)^{-1} \cdot \mathsf{pr}_2(\zeta(h))(j, i, g) = \mathsf{assoc}(h, L(r_j), L(F_{i,j,g})) \cdot \mathsf{ap}_{h\circ -}(L(K_{i,j,g})) \qquad (\mathtt{eq})$$

The problem is that this equality need not hold for wild categories.

Example 1. Define the wild category $\mathcal{E}$ by $\mathsf{Ob}(\mathcal{E}) :\equiv *$, $\hom_{\mathcal{E}}(*, *) :\equiv S^1$. The remaining structure on $\mathcal{E}$ comes from the associative H-space structure $\bullet$ on S^1. For all $\ell : \hom_{\mathcal{E}}(*, *)$, we have a nontrivial loop loop_ℓ at ℓ. Let $\Lambda : \mathcal{E} \to \mathcal{E}$ be the identity on objects and morphisms, but let $\Lambda_\circ(\ell_1, \ell_2) :\equiv \mathsf{loop}_{\ell_1 \bullet \ell_2}$. If $h \equiv \mathsf{id}_*$, it is provably false that the evident hom-adjunction $\Lambda \dashv \Lambda$ satisfies ($\mathtt{eq}$).

We isolate the following general property of an adjunction as a sufficient condition for ($\mathtt{eq}$).

Definition 2. Consider again the adjunction $L \dashv R$. We say that L is *2-coherent* if for all $h_1 : \mathsf{hom}_{\mathcal{D}}(L(X), Y)$, $h_2 : \mathsf{hom}_{\mathcal{C}}(Z, X)$, and $h_3 : \mathsf{hom}_{\mathcal{C}}(W, Z)$, the following diagram commutes:

$$\begin{array}{ccc}
(\alpha(h_1) \circ h_2) \circ h_3 & \overset{\mathsf{assoc}(\alpha(h_1), h_2, h_3)}{=\!=\!=\!=\!=} & \alpha(h_1) \circ (h_2 \circ h_3) \\
{\scriptstyle \mathsf{ap}_{-\circ h_3}(V_2(h_2, h_1))} \Vert & & \Vert {\scriptstyle V_2(h_2 \circ h_3, h_1)} \\
\alpha(h_1 \circ L(h_2)) \circ h_3 & & \alpha(h_1 \circ L(h_2 \circ h_3)) \\
{\scriptstyle V_2(h_3, h_1 \circ L(h_2))} \Vert & & \Vert {\scriptstyle \mathsf{ap}_\alpha(\mathsf{ap}_{h_1 \circ -}(L_\circ(h_2, h_3)))} \\
\alpha((h_1 \circ L(h_2)) \circ L(h_3)) & \underset{\mathsf{ap}_\alpha(\mathsf{assoc}(h_1, L(h_2), L(h_3)))}{=\!=\!=\!=\!=} & \alpha(h_1 \circ (L(h_2) \circ L(h_3)))
\end{array} \qquad (\mathtt{2\text{-}coh})$$

Theorem 3. *If L is 2-coherent, then the cocone $(L(C), L(r), L(K))$ under $L(F)$ is colimiting.*

Proof. By routine computation, ($\mathtt{eq}$) is equivalent to ($\mathtt{2\text{-}coh}$) with h_1, h_2, and h_3 instantiated with h, r_j, and $F_{i,j,g}$, respectively. ☐

Example 4. The n-truncation functor $\|-\|_n : A/\mathcal{U} \to A/\mathcal{U}$ is 2-coherent on all coslices of a universe. We have formalized this fact in Agda [6]. This example is critical to constructing colimits in categories of higher groups [2], which is examined in [4, Section 7.1].

Suspension is 2-coherent

Knowing that the suspension functor $\Sigma : \mathcal{U}^* \to \mathcal{U}^*$ preserves colimits has a few applications. For example, this would imply that *acyclic* types [1] are closed under colimits in $\mathcal{U}^*$. It also would greatly simplify the construction of stable homotopy as a homology theory [3, Corollary 2.4]. Thus, we'd like to verify that Σ is a 2-coherent left adjoint of Ω and thus preserves colimits. Here, the structure identity principle turns 2-coherence into a *homotopy of homotopy of pointed maps.*

Definition 5. Let f_1 and f_2 be pointed maps and let $(H_1, \kappa_1), (H_2, \kappa_2) : f_1 \sim_* f_2$. A *homotopy between* (H_1, κ_1) *and* (H_2, κ_2) consists of a homotopy $\mu : H_1 \sim H_2$ and a path $M_\mu : \kappa_1 =_\mu \kappa_2$ over μ.

When building the relevant homotopy of homotopies for Σ in (non-Cubical) Agda, we can derive μ from an elaborate but tractable computation. The term M_μ, however, is infeasible to construct directly. Luckily, we can avoid it by adapting *Cavallo's trick* for homogeneous types [9, Homogeneous]. In fact, we use a slightly modified notion: a pointed type is *strongly homogeneous* if it's homogeneous such that the automorphism is the identity for the basepoint.

Lemma 6. *Let $f_1, f_2 : X_1 \to_* X_2$ with X_2 strongly homogeneous. Let $(H_1, \kappa_1), (H_2, \kappa_2) : f_1 \sim_* f_2$. If $H_1 \sim H_2$, then (H_1, κ_1) and (H_2, κ_2) are homotopic.*

As loop spaces are strongly homogeneous, we avoid the infeasible construction and conclude that Σ is 2-coherent. We have formalized this fact in Agda [5].

References

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