Oracle modalities for higher dimensional types

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In [6], Swan showed that Homotopy Type Theory is a suitable setting for synthetic computability ([1, 4]). Using the lex modality ∇, which is the sheafification with respect to $\neg\neg$-stable propositions, it is possible to represent a non-computable function $A \to B$ as a function $A \to \nabla B$: such a function f is defined on a point a (written $f(a)\downarrow$) when there is an element $b : B$ such that $f(a) = \eta_B(b)$ (where $\eta_A : A \to \nabla$). Then, one constructs the modality $\bigcirc_f$ to be the smallest modality in which f is a total function, which can be done by considering the nullification (in the sens of [5]) of the family $n : \mathbb{N} \vdash f(n)\downarrow : \mathcal{U}$. The universe of $\bigcirc_f$-modal types can be seen as the smallest universe in which f is total: in fact, this point of view amounts to representing oracles as sub-toposes, which is an idea going back to [3].

The approach in [6] is mostly set-oriented, using functions to $\nabla\mathbb{N}$ or using the suspension of them. Still, HoTT is particularly suited to talk about types of higher dimension than sets, so we tried to generalize the construction of oracle modalities to families of n-types, not necessarily propositions.

In this talk, I will present such a generalization, based on CW-complexes (with a definition close to [2] but where it is allowed to construct pushouts against the same n-sphere several times), where instead of taking a function f and nullifying the family $n : \mathbb{N} \vdash f(n)\downarrow$, we take any family of types $n : \mathbb{N} \vdash A_n : \mathcal{U}$ and consider the statement that all these types are finite CW-complexes. nullification at this type family yields a new oracle construction, for which I will prove that oracle modalities are a particular case.

In a second part, I will show how this generalization can be used to define a broader notion of computably enumerable (c.e.) sets. This new notion of c.e. sets also coincides with the usual notion of c.e. set when the family of types is taken to be a family of propositions.

The final part will be a discussion about the case of families of higher dimensional types for this notion of generalized c.e. sets. For this, we study how going up in the hierarchy of n-types relates to the Turing jump, an operation which, for a function $f : \mathbb{N} \to \mathbb{N}$, associates the set of pairs $(g, x) : (\mathbb{N} \to \mathbb{N}) \times \mathbb{N}$ computable in the subuniverse associated to $\bigcirc_f$ such that $g(x)\downarrow$, written f' (for example $0'$ is the halting problem, as functions computable in 0 are computable functions). A case of interest is the case where we consider a family $n : \mathbb{N} \vdash A_n : \mathrm{Set}$. We will prove that the $0'$-c.e. sets, *i.e.* the sets which are c.e. inside the subuniverse associated to $\bigcirc_{0'}$, can be expressed as a special case of families of set which are c.e. in the generalized sense, as well as a work in progress as to how we can prove the other inclusion (that all families of sets which are c.e. are in fact $0'$-c.e. sets). This suggests that families of n-types which are c.e. in the generalized sense correspond to $0^{(n)}$-c.e. sets (the n-th iteration of the Turing jump). We will conclude with the case of 1-types to see why this may fail for higher dimension.

References

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