Towards computing the second stable homotopy group of spheres in HoTT

Tom Jack
Independent researcher, USA
pi3js2@proton.me

Axel Ljungström
Stockholm University, Sweden
axel.ljungstrom@math.su.se

Homotopy theorists have studied the homotopy groups of spheres for nearly a century [Whi83]. In 1938, Pontrjagin announced that $\pi_4\mathbb{S}^3 \cong \mathbb{Z}/2\mathbb{Z}$ [Pon38a] and that $\pi_6\mathbb{S}^4 \cong 1$ [Pon38b], but the latter was incorrect. Whitehead corrected the error in 1950, proving that $\pi_6\mathbb{S}^4 \cong \mathbb{Z}/2\mathbb{Z}$ [Whi50].[1] Here, we argue in HoTT that *at least one of them* was right: we show that $\pi_6\mathbb{S}^4 \cong \mathbb{Z}/n\mathbb{Z}$ for some $n \in \{1, 2\}$.

We proceed by constructing an exact sequence similar to Whitehead's '(metastable) EHP'[2] sequence [Whi53; DH21]. Our main theorem, Theorem 1, is a HoTT analogue of a result due to Gray [Gra73, Corollary 5.8] which allows us to refine the long exact sequence of homotopy groups associated to the so-called 'pinch' map. This gives an exact sequence (1) which appears to generalise (and extend) Whitehead's EHP sequence, though we have not proved this yet. Homotopy theorists working in spaces have recently applied related but more sophisticated methods, see [YMW24; ZJ24; Zhu24].

We work in informal 'book' HoTT [UF13]. We rely on previous computations of homotopy groups of spheres in HoTT [LS13; LB13; Bru16]. We intend to mechanise our argument in `Cubical Agda` [VMA21], relying on Ljungström and Mörtberg's recent refinement and mechanisation [LM24] of Brunerie's computation of $\pi_4\mathbb{S}^3$ [Bru16], and the rest of the 'Cubical' [Cub24] `Agda` [Agd24] library.

The basics In what follows, all types are taken to be pointed and we write $a_0 : A$ to denote basepoints. We denote the join of A and B by $A * B$ and define it as the pushout of the span $A \leftarrow A \times B \rightarrow B$. We always take it to be pointed by $\mathsf{inl}(a_0)$. Furthermore, we remind the reader that ΣA denotes the suspension of A – the higher inductive type with two point constructors $\mathsf{N}, \mathsf{S} : \Sigma A$ and a path constructor $\mathsf{merid} : A \rightarrow \mathsf{N} = \mathsf{S}$. An important map associated with suspensions of pointed types is $\sigma_A : A \rightarrow \Omega(\Sigma A)$ which we define by $\sigma_A(a) :\equiv \mathsf{merid}\,(a) \cdot \mathsf{merid}\,(a_0)^{-1}$. We will also need cofibres: cofib_f denotes the the cofibre of a map f, i.e. the pushout of the span $1 \leftarrow A \xrightarrow{f} X$. Finally, given a pointed function $f : A \xrightarrow{\bullet} B$, we write $\mathsf{ap}^{\bullet}_f : \Omega A \rightarrow \Omega B$ for the induced map on loop spaces.

The fibre of the pinch map The homotopy group computation we are planning on carrying out boils down to understanding the fibres of one particular map, $\mathsf{pinch}_f : \mathsf{cofib}_f \rightarrow \Sigma A$, called the *pinch map* of $f : A \rightarrow X$. Its definition is simple:

$$\mathsf{pinch}_f(\mathsf{inl}(*)) :\equiv \mathsf{N} \qquad \mathsf{pinch}_f(\mathsf{inr}(x)) :\equiv \mathsf{S} \qquad \mathsf{ap}_{\mathsf{pinch}_f}(\mathsf{push}\,(a)) :\equiv \mathsf{merid}\,(a)$$

The pinch map is important because it allows us to mediate between two classes of spaces at the heart of homotopy theory: (arbitrary) cofibres and suspensions. We will show that when A and X are suspensions, the fibre of the pinch map can be approximated as the cofibre of a certain *(generalised) Whitehead product* [Whi41; Ark62]. Let us recall how Brunerie [Bru16] defined these in HoTT. Given pointed maps $f : \Sigma A \xrightarrow{\bullet} C$, $g : \Sigma B \xrightarrow{\bullet} C$, we define their Whitehead product $[f, g] : A * B \xrightarrow{\bullet} C$ as follows:

$$[f, g]\,(\mathsf{inl}(a)) :\equiv c_0 \qquad [f, g]\,(\mathsf{inr}(b)) :\equiv c_0 \qquad \mathsf{ap}_{[f,g]}(\mathsf{push}(a, b)) :\equiv \mathsf{ap}^{\bullet}_g(\sigma_B(b)) \cdot \mathsf{ap}^{\bullet}_f(\sigma_A(a))$$

With this, we can state and prove our main theorem (a HoTT version of [Gra73, Corollary 5.8]).

Theorem 1. *Let $f : \Sigma A \xrightarrow{\bullet} \Sigma X$ where A and X are pointed and suppose that A is $(a-1)$-connected. There is a $2a$-connected map $\gamma : \mathsf{cofib}_\alpha \rightarrow \mathsf{fib}_p$ where α is the Whitehead product $[\mathsf{id}_{\Sigma X}, f] : X * A \xrightarrow{\bullet} \Sigma X$ and p is the pinch map $\mathsf{pinch}_f : \mathsf{cofib}_f \rightarrow \Sigma^2 A$.*

[1] Pontrjagin [Pon50] also corrected the error in 1950 [nLa25].

[2] One form of EHP sequence was constructed in HoTT by Cagne et al. [Cag+24], although it differs from ours.

Proof. We construct γ by first re-expressing its domain and codomain as pushouts. This will allow us to obtain γ from a map of spans. We consider the diagrams to the bottom right. First, the fact that square (a) is a pushout square is an easy consequence of the flattening lemma [UF13, Lemma 6.12.2]. The map W in square (b) is easy to define such that $\alpha = \mathsf{fold} \circ (\mathsf{id} \vee f) \circ W$ and we refer to [Bru16, Proposition 3.3.2] for its definition and for the fact that squares (b) and $(b)+(d)$ are pushout squares. By pushout pasting, this gives that (d) is a pushout square.[3] The fact that (c) is a pushout square is straightforward. Finally, by another pushout pasting, we find that the composite rectangle $(c)+(d)$ is a pushout square. Thus, we let γ be the map of pushouts induced by the map of spans $(\gamma_l, \gamma_m, \gamma_r)$ as described in the third diagram to the right. We define:

$$\gamma_l(x,y) :\equiv (x, \sigma_{\Sigma A}(y)) \qquad \gamma_m :\equiv \gamma_l \circ \iota^\vee \qquad \gamma_r :\equiv \sigma_{\Sigma A}$$

$$\begin{array}{ccc} \Sigma A \times \Omega\Sigma^2 A & \xrightarrow{(x,p)\mapsto \sigma_{\Sigma A}(x)\cdot p} & \Omega\Sigma^2 A \\ {\scriptstyle f\times\mathsf{id}}\downarrow & (a) & \downarrow \\ \Sigma X \times \Omega\Sigma^2 A & \longrightarrow & \mathsf{fib}_p \end{array}$$

$$\begin{array}{ccccc} & & \Sigma A \vee \Sigma A & \xrightarrow{\mathsf{fold}} & \Sigma A \\ & & {\scriptstyle f\vee\mathsf{id}}\downarrow & (c) & \downarrow{\scriptstyle f} \\ X * A & \xrightarrow{W} & \Sigma X \vee \Sigma A & \longrightarrow & \Sigma X \\ \downarrow & (b) & {\scriptstyle \iota^\vee}\downarrow & (d) & \downarrow \\ 1 & \longrightarrow & \Sigma X \times \Sigma A & \longrightarrow & \mathsf{cofib}_\alpha \end{array}$$

$$\begin{array}{ccccc} \Sigma X \times \Sigma A & \longleftarrow & \Sigma A \vee \Sigma A & \longrightarrow & \Sigma A \\ {\scriptstyle \gamma_l}\downarrow & & \downarrow{\scriptstyle \gamma_m} & & \downarrow{\scriptstyle \gamma_r} \\ \Sigma X \times \Omega\Sigma^2 A & \leftarrow & \Sigma A \times \Omega\Sigma^2 A & \rightarrow & \Omega\Sigma^2 A \end{array}$$

It follows from the Freudenthal suspension theorem [UF13, Theorem 8.6.4] that $\sigma_{\Sigma A} : \Sigma A \to \Omega\Sigma^2 A$ is $2a$-connected (using that ΣA is a-connected). Furthermore, the wedge inclusion $\iota^\vee$ appearing in the definition of γ_m is $2a$-connected [UF13, Lemma 8.6.2]. It follows that γ_l, γ_m and γ_r all are $2a$-connected and, hence, so is the induced map $\gamma : \mathsf{cofib}_\alpha \to \mathsf{fib}_p$. □

Now, let p and α be as in Theorem 1 and let A be $(a-1)$-connected and X be $(x-1)$-connected. Let us investigate what happens when we instantiate the long exact sequence of homotopy groups [UF13, §8.4] with the fibration sequence $\mathsf{fib}_p \to \mathsf{cofib}_f \to \Sigma A$. We obtain the long exact sequence

$$\cdots \to \pi_{n+1}(\Sigma^2 A) \xrightarrow{\partial_n} \pi_n(F_n) \to \pi_n(\mathsf{cofib}_f) \xrightarrow{p_*} \pi_n(\Sigma^2 A) \xrightarrow{\partial_{n-1}} \pi_{n-1}(F_{n-1}) \to \dots \qquad (1)$$

where $\qquad F_{n\leq x+a} :\equiv \Sigma X \qquad F_{x+a<n\leq 2a} :\equiv \mathsf{cofib}_\alpha \qquad F_{2a<n} :\equiv \mathsf{fib}_p.$

The above sequence follows by replacing fib_p by cofib_α in low enough dimensions, using Theorem 1. The reason we can in turn replace cofib_α by ΣX in the 'metastable range' $n \leq x + a$ is an easy consequence of the connectedness of X.

Application to homotopy groups of spheres Let us put our sequence to the test. It gives us a characterisation of the unstable homotopy group $\pi_5\mathbb{S}^3$, and therefore also of the stable homotopy group $\pi_6\mathbb{S}^4$ (the fact that these coincide follows from the quaternionic Hopf fibration [BR18, Theorem 4.10]).

Corollary 1. *There is an integer $n \in \{1,2\}$ such that $\pi_5\mathbb{S}^3 \cong \pi_6\mathbb{S}^4 \cong \mathbb{Z}/n\mathbb{Z}$.*

Proof. We instantiate (1) with $A :\equiv \mathbb{S}^2$, $X :\equiv \mathbb{S}^1$, $f :\equiv [\mathsf{id}_{\mathbb{S}^2}, \mathsf{id}_{\mathbb{S}^2}] : \mathbb{S}^3 \to \mathbb{S}^2$, $a = 2$ and $x = 1$. In this situation, we have a pretty good understanding of the spaces appearing in (1). **First**, Brunerie [Bru16] showed that $\mathsf{cofib}_f \simeq \mathsf{J}_2\mathbb{S}^2$, where $\mathsf{J}_2\mathbb{S}^2$ denotes the *second James construction on* $\mathbb{S}^2$ – a type which, in particular, satisfies $\pi_n\mathsf{J}_2\mathbb{S}^2 \cong \pi_{n+1}\mathbb{S}^3$ for $n \in \{3,4\}$. **Second**, we know what cofib_α is: indeed, we have $\alpha :\equiv [\mathsf{id}_{\mathbb{S}^2}, [\mathsf{id}_{\mathbb{S}^2}, \mathsf{id}_{\mathbb{S}^2}]]$. Now, we know that $[\mathsf{id}_{\mathbb{S}^2}, \mathsf{id}_{\mathbb{S}^2}] = \pm 2h$, where $h : \mathbb{S}^3 \to \mathbb{S}^2$ is the Hopf map (i.e. the generator of $\pi_3\mathbb{S}^2$ [Bru16; LM24]). Using bilinearity of the Whitehead product (proved in unpublished work by Ljungström [Lju24]), we get $\alpha = [\mathsf{id}_{\mathbb{S}^2}, \pm 2h] = \pm 2[\mathsf{id}_{\mathbb{S}^2}, h]$ which must vanish in $\pi_4\mathbb{S}^2$ due to 2-torsion. Thus, cofib_α is the pushout along a constant map. This gives us an equivalence $\mathsf{cofib}_\alpha \simeq \mathbb{S}^2 \vee \mathbb{S}^5$ (which agrees with $\mathbb{S}^2$ on π_4). Modulo these characterisations of cofib_f and cofib_α, our instance of (1) provides us with the following exact sequence.

$$\pi_5\mathbb{S}^4 \xrightarrow{\partial_4} \pi_4\mathbb{S}^2 \to \pi_5\mathbb{S}^3 \to \pi_4\mathbb{S}^4 \xrightarrow{\partial_3} \pi_3\mathbb{S}^2 \to \pi_4\mathbb{S}^3 \to 1 \qquad (2)$$

By nature of the groups involved, the homomorphism $\partial_3 : \pi_4\mathbb{S}^4 \to \pi_3\mathbb{S}^2$ must be either injective or 0. If it were 0, then (2) would give us an injection $\pi_3\mathbb{S}^2 \to \pi_4\mathbb{S}^3$ which is impossible due to 2-torsion in the codomain. So ∂_3 is injective. This means that the second map in (2) is surjective, which implies that $\pi_5\mathbb{S}^3 \cong \pi_4\mathbb{S}^2/\mathsf{Im}\,(\partial_4)$. The statement then follows immediately from the isomorphisms $\pi_5\mathbb{S}^4 \cong \pi_4\mathbb{S}^2 \cong \mathbb{Z}/2\mathbb{Z}$ [Bru16]. □

Yet another Brunerie number... Fortunately, we do not expect it to be too difficult to prove $n = 2$. We conjecture that trivialising a certain Whitehead product should suffice, and there are direct approaches to this in HoTT [LM24, §6] – but that is a topic for a future talk.

References

[Agd24] Agda Developers. *Agda*. Version 2.8.0. 2024. URL: https://agda.readthedocs.io/.

[Ark62] M. Arkowitz. "The generalized Whitehead product." In: *Pacific Journal of Mathematics* 12.1 (1962), pp. 7–23. DOI: 10.2140/pjm.1962.12.7.

[Bru16] G. Brunerie. "On the homotopy groups of spheres in homotopy type theory". PhD thesis. Université Nice Sophia Antipolis, 2016. URL: http://arxiv.org/abs/1606.05916.

[BR18] U. Buchholtz and E. Rijke. "The Cayley-Dickson Construction in Homotopy Type Theory". In: *Higher Structures* 2.1 (2018), pp. 30–41. DOI: 10.21136/HS.2018.02.

[Cag+24] P. Cagne et al. "On symmetries of spheres in univalent foundations". In: *Proceedings of the 39th Annual ACM/IEEE Symposium on Logic in Computer Science*. LICS '24. Tallinn, Estonia: Association for Computing Machinery, 2024. ISBN: 9798400706608. DOI: 10.1145/3661814.3662115. URL: https://doi.org/10.1145/3661814.3662115.

[Cub24] The Agda Community. *Cubical Agda Library*. Version 0.7. Feb. 2024. URL: https://github.com/agda/cubical.

[DH21] S. Devalapurkar and P. Haine. "On the James and Hilton-Milnor splittings, and the metastable EHP sequence". In: *Documenta Mathematica* 26 (2021), pp. 1423–1464. ISSN: 1431-0643. DOI: 10.4171/dm/845. URL: http://dx.doi.org/10.4171/dm/845.

[Gra73] B. Gray. "On the Homotopy Groups of Mapping Cones". In: *Proceedings of the London Mathematical Society* s3-26.3 (Apr. 1973), pp. 497–520. ISSN: 0024-6115. DOI: 10.1112/plms/s3-26.3.497. eprint: https://academic.oup.com/plms/article-pdf/s3-26/3/497/4240226/s3-26-3-497.pdf. URL: https://doi.org/10.1112/plms/s3-26.3.497.

[LB13] D. R. Licata and G. Brunerie. "$\pi_n(S^n)$ in Homotopy Type Theory". In: *Certified Programs and Proofs*. Ed. by G. Gonthier and M. Norrish. Cham: Springer International Publishing, 2013, pp. 1–16. ISBN: 978-3-319-03545-1.

[LS13] D. R. Licata and M. Shulman. "Calculating the Fundamental Group of the Circle in Homotopy Type Theory". In: *Proceedings of the 2013 28th Annual ACM/IEEE Symposium on Logic in Computer Science*. LICS '13. New Orleans, LA, USA: IEEE Computer Society, 2013, pp. 223–232. ISBN: 978-0-7695-5020-6. DOI: 10.1109/LICS.2013.28.

[Lju24] A. Ljungström. *Formalisation of super Lie algebra properties of Whitehead products*. 2024. URL: https://github.com/agda/cubical/pull/1170.

[LM24] A. Ljungström and A. Mörtberg. *Formalising and Computing the Fourth Homotopy Group of the 3-Sphere in Cubical Agda*. 2024. arXiv: 2302.00151 [math.AT]. URL: https://arxiv.org/abs/2302.00151.

[nLa25] nLab authors. *second stable homotopy group of spheres*. https://ncatlab.org/nlab/show/second+stable+homotopy+group+of+spheres. Revision 8. Feb. 2025.

[Pon38a] L. Pontrjagin. "Classification of continuous maps of a complex into a sphere, Communication I". In: *Doklady Akademii Nauk SSSR* 19.3 (1938), pp. 147–149.

[Pon38b] L. Pontrjagin. "Classification of continuous maps of a complex into a sphere, Communication II". In: *Doklady Akademii Nauk SSSR* 19.5 (1938), pp. 361–363.

[Pon50] L. Pontrjagin. "Homotopy classification of mappings of an (n+2)-dimensional sphere on an n-dimensional one". In: *Doklady Akademii Nauk SSSR* 70.6 (1950), pp. 957–959.

[UF13] The Univalent Foundations Program. *Homotopy Type Theory: Univalent Foundations of Mathematics*. Institute for Advanced Study: Self-published, 2013. URL: https://homotopytypetheory.org/book/.

[VMA21] A. Vezzosi, A. Mörtberg, and A. Abel. "Cubical Agda: A Dependently Typed Programming Language with Univalence and Higher Inductive Types". In: *Journal of Functional Programming* 31 (2021), e8. DOI: 10.1017/S0956796821000034.

[Whi83] G. W. Whitehead. "Fifty years of homotopy theory". In: *Bulletin of the American Mathematical Society* 8.1 (1983), pp. 1–29.

[Whi50] G. W. Whitehead. "The (n+ 2) nd homotopy group of the n-sphere". In: *Annals of Mathematics* 52.2 (1950), pp. 245–247.

[Whi53] G. W. Whitehead. “On the Freudenthal Theorems”. In: *Annals of Mathematics* 57.2 (1953), pp. 209–228. ISSN: 0003486X, 19398980. URL: http://www.jstor.org/stable/1969855 (visited on 01/08/2025).

[Whi41] J. H. C. Whitehead. “On Adding Relations to Homotopy Groups”. In: *Annals of Mathematics* 42.2 (1941), pp. 409–428. ISSN: 0003486X, 19398980. URL: http://www.jstor.org/stable/1968907 (visited on 01/14/2025).

[YMW24] J. Yang, J. Mukai, and J. Wu. *On the Homotopy Groups of the Suspended Quaternionic Projective Plane and Applications*. 2024. arXiv: 2301.06776 [math.AT]. URL: https://arxiv.org/abs/2301.06776.

[Zhu24] Z. Zhu. *The unstable homotopy groups of 2-cell complexes*. 2024. arXiv: 2410.20416 [math.AT]. URL: https://arxiv.org/abs/2410.20416.

[ZJ24] Z. Zhu and T. Jin. “The relative James construction and its application to homotopy groups”. In: *Topology and its Applications* 356 (2024), p. 109043. ISSN: 0166-8641. DOI: https://doi.org/10.1016/j.topol.2024.109043. URL: https://www.sciencedirect.com/science/article/pii/S0166864124002281.