# Progress report on constructive higher presheaf models of HoTT

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We present ongoing work on a number of constructions related to higher presheaf models of homotopy type theory in a weak and in particular constructive and predicative meta theory. This continues the line of work by Coquand, Ruch, and Sattler [3]. In particular, we present constructions related to obtaining presheaf models over categories internal to cubical sets, and presheaf models with dependent choice. We hope to combine these two approaches, to constructively justify the synthetic study of light condensed sets by Cherubini et al. [1].

## 1 Presheaf models over internal categories

One goal is to generalize the constructions by Coquand, Ruch, and Sattler [3] to internal categories. One application of this work is the constructive study of arbitrary higher presheaves, not just those over strict 1-categories. Another motivation is the following mismatch, that occurs when working constructively.

Let $\mathbb{C}$ be a concrete category, so a category of structured sets $(X, x), (Y, y), \ldots$ (e.g. groups) and functions between them. As described by Coquand, Ruch, and Sattler [3], one obtains a model of HoTT on $\widehat{\mathbb{C}^{\mathrm{op}} \times \square} \cong \mathbb{C} \to \mathbf{cSet}$. A closed type in this model is a presheaf $A : \mathbb{C} \to \mathbf{cSet}$ that has levelwise the structure of a fibrant cubical set, for which all restriction maps are morphisms of these structures.

Often, one wants to define presheaves pointwise in terms of objects of the base category, such as the presheaf given at $(X, x)$ by the underlying set $X$ as a discrete cubical set. In the case where $\mathbb{C}$ is an algebraic category, this object is representable. We would like this object to remain representable when taking certain quotients. Performing the construction pointwise in **Set** using strict quotients yields presheaves which constructively do not have the correct universal property with respect to all h-sets. Furthermore, a weak metatheory might not have strict quotients available. Performing the construction pointwise in cubical sets using homotopy quotients yields a type with the correct universal property. However, in general, the homotopy quotient is representable up to equivalence only in the presence of choice.

We resolve this mismatch by instead replacing the objects of $\mathbb{C}$ by structured h-sets. This forces us to pass to a category *internal* to cubical sets. By adapting the strictification construction by Coquand, Ruch, and Sattler [3], we can strictify h-set-valued presheaves homotopy coherent presheaves, to ordinary fibrant presheaves over this category. Assuming a fibrant strict category, this enables a transfer of certain constructions from presheaves over it (internal to the model of HoTT), to the strict and fibrant presheaves that form the presheaf model of HoTT.

## 2 Presheaf models with dependent choice

Another goal of the ongoing work is the generalization of a forthcoming model construction by Sattler [4, 2] to presheaves. This model presents $\infty$-groupoids, which entails the following pointwise principle.

**Theorem 1.** For $P \in \mathrm{Ty}(\Gamma)$ an h-prop, there is a map $\mathrm{El}_{\mathbf{Set}}(\Gamma_0, P_0) \to \mathrm{El}_{\widehat{\square}}(\Gamma, P)$ natural in $\Gamma$.

As a corollary of this principle, and the fact that a type and its propositional truncation agree on points, one obtains witnesses of countable and dependent choice.

We generalize the above construction to cubical presheaves over an arbitrary category $\mathbb{C}$ and obtain the analogous statement.

**Theorem 2.** For $P \in \mathrm{Ty}(\Gamma)$ an h-prop, there is a map $\mathrm{El}_{\widehat{\mathbb{C}}}(\Gamma_0, P_0) \to \mathrm{El}_{\widehat{\square \times \mathbb{C}}}(\Gamma, P)$ nat. in $\Gamma$.

From this we derive the same principles for cubical presheaf models over a category $\mathbb{C}$. Currently, we are looking into generalizing this construction to internal categories as-well.

## References


[1] Felix Cherubini et al. *A Foundation for Synthetic Stone Duality*. 2024. arXiv: 2412.03203 [math.LO]. URL: https://arxiv.org/abs/2412.03203.

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[3] Thierry Coquand, Fabian Ruch, and Christian Sattler. "Constructive sheaf models of type theory". In: *Math. Structures Comput. Sci.* 31.9 (2021), pp. 979–1002. ISSN: 0960-1295,1469-8072. DOI: 10.1017/S0960129521000359. URL: https://doi.org/10.1017/S0960129521000359.

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