The algebraic small object argument as a saturation

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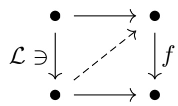
We describe work in progress on a principle for weak factorization systems (WFS's) generated by Garner's algebraic small object argument which mirrors the characterization of left maps in a WFS permitting Quillen's small object argument: as the least "saturated" class containing the generators.

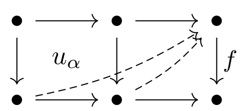
Factorization systems A WFS on a category $\mathcal{E}$ consists of classes of "left" maps $\mathcal{L} \subseteq \mathcal{E}^{\to}$ and "right" maps $\mathcal{R} \subseteq \mathcal{E}^{\to}$ such that (a) every map in $\mathcal{E}$ factors as a left followed by a right map and (b) each class is characterized by a *lifting property*: f is a right map if and only if in every square



{lifting} (1)

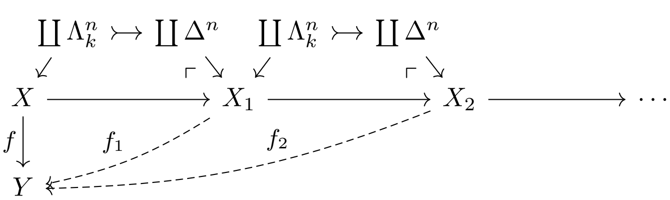
there exists a dashed lift as indicated, and a map is a left map if and only if it has the dual lifting property against right maps. Homotopical semantics of type theory have been tied to the concept of WFS since their earliest days [AW09]: types $\Gamma \vdash A$ are interpreted as right maps $A \to \Gamma$, the identity family on $A \to \Gamma$ is interpreted by factorizing the diagonal $A \to A \times_\Gamma A$, and the identity elimination rule comes from lifting.

In practice a WFS is often *generated* by a set of basic left maps, with the right maps taken to be those which lift against the generating maps; the left maps are those lifting against all right maps, which includes the generators. For example, the Kan (trivial cofibration, fibration) WFS that underlies the simplicial model [KL21] is generated by the *horn inclusions* $\Lambda^n_k \rightarrowtail \Delta^n$. Garner [Gar09] introduced the concept of a WFS generated by a *diagram* $u\colon \mathcal{J} \to \mathcal{E}^{\to}$ of arrows in the category: a right map is one that can be assigned lifts in each problem as in (1) against a map of the form u_i in such a way that the resulting triangles



commute for all $\alpha\colon j \to i$. Such WFS's appear in cubical set semantics of HoTT [GS17; CMS20; Awo23; ACCRS24] where types are maps lifting "uniformly" against open boxes. In some cases, these WFS's can be generated by a set classically but not constructively; in others, such as Cohen et al.'s "face lattice" definition [CCHM15], we expect that generation from a set is impossible even in a classical metatheory.

Small object arguments To actually produce a WFS from a set or diagram of generators, we must be able to factorize any map as a left map followed by a right map. The classical tool is Quillen's *small object argument* [Qui67], named for a condition that the generators are in some sense "small". One thinks of factorizing a map f as approximating it by a right map, which we can do by iteratively attaching solutions to every lifting problem against a generator to its domain, as pictured here for the case of horn inclusions:



{complex-process} (2)

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Note that attaching solutions to lifting problems can create new lifting problems. Under the aforementioned smallness condition, however, iterating up to a sufficiently large ordinal κ produces a right map $f_\kappa \colon Y_\kappa \to X$. The composite map $i_\kappa \colon X \to X_\kappa$ can be shown to be a left map, so that $f = f_\kappa i_\kappa$ is the desired factorization.

Garner's *algebraic small object argument* [Gar09; BG16a] is a refinement of Quillen's that can generate WFS's from diagrams as well as sets. It produces a sequence much like (2), with a crucial difference: when a solution to a lifting problem is added that was already added at a previous stage, the two are identified. Garner's argument smooths over quirks of Quillen's: it derives from a standard free monad construction and the transfinite iteration actually converges. Moreover, one gets an *algebraic* WFS [GT06], in which it is possible to work with left and right maps as *structured maps* rather than maps with a *property*.

Saturation Despite the algebraic small object argument's good properties, it lacks one convenience of Quillen's argument: a combinatorial description of the left maps. From (2), we see that the left factor of Quillen's factorization is a transfinite composite of pushouts (along arbitrary maps) of coproducts of generators. A map admitting such a presentation is called a *cell complex*, and it is a corollary that any left map is a retract of a cell complex. One says that a class closed under coproducts, pushouts, transfinite composites, and retracts is *saturated*. The class of left maps of a WFS is always saturated, so the left class of a WFS generated by Quillen's argument is the *least saturated class* containing the generators. This characterization can be used to reduce checking properties of left maps—*e.g.*, that a functor preserving coproducts, pushouts, and transfinite composites preserves them—to the case of generators.

This picture is less simple in Garner's argument. While the left factor *is* a transfinite composite as in (2), a consequence of quotienting is that the maps $X_i \to X_{i+1}$ may not be left maps. Thus, while Athorne [Ath12; Ath14] manages to generalize the cell complex description to some instances of WFS's generated by diagrams, we lack a description of left factors as built from simple operations *on left maps*.

The algebraic small object argument as a saturation We present an analogue of the saturation principle for Garner's small object argument using the algebraic view of left maps as structured maps. The key observation is that while the transition maps $X_i \to X_{i+1}$ may not be left maps, the maps $X \to X_i$ from the original domain to a stage are left maps. Moreover, when we remember the left map *structure* associated to each of these maps, the diagram

$$\begin{array}{ccccccccc} X & = & X & = & X & = & \cdots & = & X \\ \downarrow & & \downarrow & & \downarrow & & & & \downarrow \\ X & \longrightarrow & X_1 & \longrightarrow & X_2 & \longrightarrow & \cdots & \longrightarrow & X_\kappa \end{array}$$

describes the left factor $X \to X_\kappa$ as a colimit in **LMap**, the *category of maps equipped with left structure*. The pushouts appearing in (2) have a similar analogue in Garner's argument: one uses the closure of **LMap** under certain pushouts. The role played by coproducts of generators in (2) is played by maps in the image of *density comonad* $\mathrm{Lan}_u\, u \colon \mathcal{E}^{\to} \to \mathcal{E}^{\to}$ [Koc66] of the generating diagram $u \colon \mathcal{J} \to \mathcal{E}^{\to}$, which factors through **LMap** and can be calculated for concrete diagrams such as that of open boxes used in homotopical semantics. Finally, the argument requires that we can *compose* left map structures on composable maps; this property is captured well by the Bourke and Garner's [BG16a; BG16b; Bou23] double-categorical description of algebraic WFS's, where the maps equipped with left structure are the vertical morphisms of a *double category* $\mathbb{L}\mathbf{Map} \to \mathbb{S}\mathrm{q}(\mathcal{E})$ over the double category of squares in $\mathcal{E}$.

With this analysis, we are able to show an analogue of the saturation principle: given a notion of structured map in $\mathcal{E}$ encoded as a suitable double category $\mathbb{A} \to \mathbb{S}\mathrm{q}(\mathcal{E})$ whose "$\mathbb{A}$-structures" are closed under the colimits sketched above, we show that any lift of the density comonad $\mathrm{Lan}_u\, u$ of the generating diagram $u \colon \mathcal{J} \to \mathcal{E}^{\to}$ through the category of vertical morphisms in $\mathbb{A}$ yields a lift $\mathbb{L}\mathbf{Map} \to \mathbb{A}$ assigning an $\mathbb{A}$-structure to every left map. Note that **LMap** itself is closed under *all* colimits, but we benefit from recognizing that its objects can be built from generators using only colimits of the forms above.

Our original motivation for this work was a "routine" one: to checking that certain functors preserve the trivial cofibrations of certain model structures on cubical sets built using Garner's small object argument. However, we also believe that this technique can be used to give a constructive and algebraic proofs of the *fibration extension property* [Sat17, §7], part of a recipe for building model structures from homotopical models of type theory, without relying on a universe as in the current state of the art [Awo23; ACCRS24].

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