Propositional Geometric Type Theory

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We introduce Propositional Geometric Type Theory, an extension of Martin-Löf Type Theory (MLTT) in which types correspond to generalised classifying spaces of propositional geometric theories.

Motivation Grothendieck toposes can be seen as generalised spaces [5, 6], as well as well-behaved categories which can serve as a semantic setting for MLTT [4]. A geometric theory is a collection of axioms in a restriction of infinitary first order logic. It is well-known that toposes classify geometric theories in the sense that for every geometric theory $\mathbb{T}$ there is a topos $[\mathbb{T}]$ such that models of $\mathbb{T}$ in an arbitrary topos $\mathcal{E}$ correspond to geometric morphisms of type $\mathcal{E} \to [\mathbb{T}]$. Conversely, every topos classifies a geometric theory [2]. The ubiquity of toposes in various areas of mathematics, as well as their dual nature of being both geometric and logical objects make them worthwile objects of study.

The interaction between the internal type theory of a topos and the theory it classifies together with the geometric morphisms it generates is currently not sufficiently well understood. This limits the extent to which we can transfer results from one topos to another in a more parametric way than the bridge technique [2] permits. Diagrams of toposes can be studied using multimodal adjoint type theory [8]; it is the fact that a general geometric morphism does not preserve Π-types and that the category of toposes naturally forms a $(2,2)$-category, that makes this a rather technical undertaking.

We expect that a good theory of classifying spaces as syntactic types can overcome these challenges. Such a theory has been suggested and approached in [12, 11] using encodings of over-toposes via GRD-systems or considering the more general arithmetic universes, and in [10] by parametrising internal type theories.

In this work we restrict our attention to *propositional* geometric theories, i.e., those for which the underlying first-order signature only uses propositional symbols. Their classifying toposes are exactly those that can be presented as categories of sheaves on a locale. We consider a completion of the category of localic toposes and use the Sierpiǹski space to relate the internal language of a space to its relationship with other spaces. We expect that our work can be extended to the setting of general higher toposes.

Syntax We extend standard MLTT with a type $\mathbf{S}$, the *Sierpiǹski type*. It comes equipped with the universal open and closed points $\check{0} : \mathbf{S}$ and $\check{1} : \mathbf{S}$, respectively. We also assume a greatest lower bound operation $\wedge : \mathbf{S} \times \mathbf{S} \to \mathbf{S}$ satisfying the obvious commutative monoid laws. The Sierpiǹski type classifies *open subtypes*,

$$\frac{\gamma : \Gamma \vdash p_\gamma : \mathbf{S}}{\gamma : \Gamma \vdash \mathbf{T}(p_\gamma)\ \mathrm{tp}},$$

corresponding to the (generalised) propositional geometric theory classified by Γ in which p_γ holds. This setup lets us recover locale theory synthetically. For instance, we gain access to inverse image maps, the specialisation order, and a definition of *overt types* I, which can serve as indexing types for the infinitary disjunctions of geometric logic.

Semantics We view the category of locales as a 1-category, forgetting about the poset-enrichment. This category is not locally Cartesian closed, so we endow it with the subcanonical open cover topology and interpret our theory in the resulting sheaf topos.[1] The Sierpiǹski type is interpreted as the (Yoneda embedding of) the Sierpiǹski locale, and open subtypes arise as pullback along the open point:

$$\begin{array}{ccc} \mathbf{T}(p) & \longrightarrow & \mathbf{1} \\ \big\downarrow & \lrcorner & \big\downarrow \check{\mathrm{i}} \\ \Gamma & \xrightarrow{\;p\;} & \mathbf{S} \end{array} .$$

Future directions As a tool for synthetic locale theory, propositional geometric type theory has various competitors [9, 3]. We view it as a stepping stone for the definition of a univalent geometric type theory, with semantics in (a completion of) the category of $(\infty, 1)$-toposes.

While in our theory the Sierpiǹski type acts as a dualising object translating between terms and open subtypes, a prominent role in the full theory will be served by the theory $\mathbf{O}$ of an object with the property that terms of type $A \to \mathbf{O}$ correspond to sheaves on A or étale spaces over A.

References

[1] Clark Barwick and Peter Haine. 2019. Pyknotic objects, I. Basic notions. (2019). arXiv: 1904.09966 [math.AG].

[2] Olivia Caramello. 2017. *Theories, Sites, Toposes: Relating and studying mathematical theories through topos-theoretic 'bridges'*. Oxford University Press, (Dec. 2017). doi:10.1093/oso/9780198758914.001.0001.

[3] Martín Escardó. 2004. Synthetic topology: of data types and classical spaces. *Electronic Notes in Theoretical Computer Science*, 87, 21–156. Proceedings of the Workshop on Domain Theoretic Methods for Probabilistic Processes. doi:https://doi.org/10.1016/j.entcs.2004.09.017.

[4] Martin Hofmann. 1997. *Syntax and semantics of dependent types*. Springer London, London, 13–54. doi:10.1007/978-1-4471-0963-1_2.

[5] Peter T Johnstone. 2002. *Sketches of an Elephant A Topos Theory Cornpendium*. Oxford University Press, (Sept. 2002). ISBN: 9780198534259. doi:10.1093/oso/9780198534259.001.0001.

[6] Saunders Mac Lane and Ieke Moerdijk. 1994. *Sheaves in Geometry and Logic*. Springer, New York. doi:10.1007/978-1-4612-0927-0.

[7] Peter Scholze. 2020. Lectures on condensed mathematics. All results joint with Dustin Clausen. (2020). https://www.math.uni-bonn.de/people/scholze/Condensed.pdf.

[8] Michael Shulman. 2023. Semantics of multimodal adjoint type theory. *Electronic Notes in Theoretical Informatics and Computer Science*, (Nov. 2023). doi:10.46298/entics.12300.

[9] Paul Taylor. 2000. Geometric and higher order logic in terms of abstract stone duality. eng. *Theory and Applications of Categories [electronic only]*, 7, 284–338. http://eudml.org/doc/121233.

[10] Taichi Uemura. [n. d.] Synthetic topos theory. https://uemurax.github.io/synthetic-topos-theory/. Accessed: 2025-02-07. ().

[11] Steven Vickers. 2017. Arithmetic universes and classifying toposes. (2017). arXiv: 1701.04611 [math.CT].

[12] Steven Vickers. 2014. Continuity and geometric logic. *Journal of Applied Logic*, 12, 1, 14–27. doi:10.1016/j.jal.2013.07.004.

[1] The category of locales is not small, so there is a bit of a size issue, analogous to what happens when one wants to take sheaves on the large category of compact Hausdorff spaces. We can deal with this size issue in either of two standard ways: (1) we can restrict to small locales, as in the pyknotic approach of [1], or (2) we can restrict the sheaves, as in the condensed approach of [7].