The Yoneda embedding in simplicial type theory

Daniel Gratzer[1], Jonathan Weinberger[2], and Ulrik Buchholtz[3]

[1] Department of Computer Science, Aarhus University
[2] Fowler School of Engineering, Schmid College of Science and Technology, and Center of Excellence in Computation, Algebra, and Topology (CECAT), Chapman University
[3] School of Computer Science, University of Nottingham

In previous work [1] we showed how to define the category of spaces $\mathcal{S}$ in the simplicial type theory of Riehl and Shulman [2] by moving to a modal type theory and working in the larger model of cubical spaces. Now we'd like to report on more recent work [3], in which we add a twisted arrow modality in order to construct the Yoneda embedding. We then take advantage of this in order to deduce a host of classical results, including:

- that pointwise invertible maps in $C \to D$ are invertible;
- that pointwise left adjoints are left adjoints;
- that (co)limits are computed pointwise in $C \to D$;
- the theory and existence of pointwise Kan extensions;
- Quillen's theorem A;
- the properness of cocartesian fibrations (cf. [4]).

Our synthetic approach gives proofs of comparative complexity to those in 1-category theory, while applying to $(\infty, 1)$-categories, yielding a drastic reduction in technicality. It has parallels to the synthetic higher category theory of [5], though in our theory not every type is a category. A consequence of this is that we are able to prove some of their axioms to be valid in our theory. Though our theory lacks a number of the results that [5] have established we are optimistic that, pending further investigations and possibly extensions, we will eventually be able to reproduce most of their results in our theory as well, at least insofar as they pertain to the "category-theoretic fragment."

Let us now describe in more detail the setting, the axioms regarding twisted arrows, and some of the results obtained.

Setting Simplicial type theory allows us to do synthetic category theory by postulating a directed interval type $\mathbb{I}$, a bounded distributive lattice $(0, 1, \vee, \wedge)$ such that $\prod_{i,j:\mathbb{I}} i \leq j \vee j \leq i$ holds. The synthetic morphisms in a type X are then given by maps $\mathbb{I} \to X$. From $\mathbb{I}$ we can define the n-cubes $\mathbb{I}^n$, from which we isolate simplices Δ^n, boundaries $\partial\Delta^n$, and horns Λ^n_k. In particular, $\Delta^2 \to X$ represents an 2-cell in X witnessing the composition of two arrows, and $\Lambda^2_1 \to X$ represents a pair of composable arrow (without a composite).

Definition 1. A *precategory* is a type X satisfying the Segal condition: $\mathsf{isEquiv}(X^{\Delta^2} \to X^{\Lambda^2_1})$. A *category* is a precategory X satisfying the Rezk condition: $\prod_{x,y:C} \mathsf{isEquiv}((x = y) \to \mathrm{iso}(x, y))$, where $\mathrm{iso}(x, y)$ is the type of isomorphisms from x to y (i.e., admitting left and right inverses).

The theory of adjunctions in this setting has been developed already by Riehl and Shulman [2] and that of (co)limits by Bardomiano Martínez [6]. To get any useful examples of categories, we constructed in [1] a subuniverse $\mathcal{S}$ satisfying, among other things:

- If $X : A \to \mathcal{S}$, then the composite $A \to \mathcal{U}$ is covariant, as defined in [2].
- The converse holds for $A :_\flat \mathcal{U}$, $X :_\flat A \to \mathcal{U}$: if X is covariant, then X factors through $\mathcal{S}$.

Notice that we here make use of (multi)modal type theory, MTT [7, 8] to restrict in the converse direction to *crisp* families, i.e., with no implied functoriality. This is similar to the use of spatial type theory [9] to allow quantification over topological ∞-groupoids with no implied continuity.

We use a mode theory with a single mode for simplicial spaces and the following modalities in addition to $\flat$: $\sharp$ (the right adjoint to $\flat$), op (the opposite simplicial space), and tw (twisted arrows, see below), subject to the following (in)equalities:

$$\flat \circ \flat = \flat \circ \mathsf{op} = \flat \circ \sharp = \flat \qquad \sharp \circ \sharp = \sharp \circ \mathsf{op} = \sharp \circ \flat = \sharp \qquad \mathsf{op} \circ \mathsf{op} = \mathsf{id} \qquad \flat \leq \mathsf{id} \leq \sharp \qquad \flat \leq \mathsf{tw}$$

Each modality μ acts on types X, written $\langle \mu \mid X \rangle$. Intuitively, for crisp categories C, the actions of the modalities can be described as follows: $\langle \flat \mid C \rangle$ is the space of points of C; $\langle \sharp \mid C \rangle$ is the codiscretization of C, where there is exactly one arrow between each pair of points; $\langle \mathsf{op} \mid C \rangle$ is the opposite category of C.

The twisted arrow modality If C is a crisp category, we may form the opposite category $\langle \mathsf{op} \mid C \rangle$, and hence the presheaf category $\widehat{C} := \mathcal{S}^{\langle \mathsf{op} \mid C \rangle}$. However, we need an extra ingredient to form the Yoneda embedding $\mathbf{y} : C \to \widehat{C}$. Indeed, taking synthetic morphisms gives a map $\hom(-,-) : C \times C \to \mathcal{U}$ that doesn't factor through $\mathcal{S}$. What is required instead is a function $\Phi : \langle \mathsf{op} \mid C \rangle \times C \to \mathcal{S}$ such that $\Phi(\mathsf{mod}_{\mathsf{op}}(c), -) = \hom(c, -)$ whenever $c :_\flat C$, i.e., a function that agrees on objects with $\hom(-,-)$ and has the same functoriality in the second argument, but takes $\langle \mathsf{op} \mid C \rangle$, introduced via $\mathsf{mod}_{\mathsf{op}}()$, as its first argument. This is where tw comes in. We axiomatize this such that the space of n-simplices in $\langle \mathsf{tw} \mid C \rangle$, $\langle \flat \mid \Delta^n \to \langle \mathsf{tw} \mid C \rangle \rangle$, can be visualized as follows:

$$\begin{array}{ccccccc} c_n & \longleftarrow & c_{n-1} & \longleftarrow & \cdots & \longleftarrow & c_0 \\ \downarrow & & & & & & \\ c_{n+1} & \longrightarrow & c_{n+2} & \longrightarrow & \cdots & \longrightarrow & c_{2n} \end{array} \tag{1}$$

That is, we get a *covariant* function $\langle \mathsf{tw} \mid C \rangle \to \langle \mathsf{op} \mid C \rangle \times C$, and the induced map $\langle \mathsf{op} \mid C \rangle \times C \to \mathcal{S}$ is the desired function Φ. Then we define $\mathbf{y} := \lambda c.\, \Phi(-, c)$ to be the Yoneda embedding. The analogy of $\langle \mathsf{tw} \mid C \rangle$ with the usual arrow category is as follows: while arrows in $\mathrm{Arr}(C) := C^{\mathbb{I}}$ are squares as usual (left), arrows in $\langle \mathsf{tw} \mid C \rangle$ are *twisted squares* (right):

$$\begin{array}{ccc} a & \longrightarrow & a' \\ \downarrow & & \downarrow \\ b & \longrightarrow & b' \end{array} \qquad\qquad \begin{array}{ccc} a & \longleftarrow & a' \\ \downarrow & & \downarrow \\ b & \longrightarrow & b' \end{array}$$

Some selected results The first result is the functorial Yoneda lemma, improving on the version in [2]: There is a natural isomorphism $\Phi_{\widehat{C}}(\mathbf{y}^\dagger(-), -) \cong \mathsf{eval} : \langle \mathsf{op} \mid C \rangle \times \widehat{C} \to \mathcal{S}$. Here we use both Φ_C, in the guise of the opposite of $\mathbf{y}$, $\mathbf{y}^\dagger : \langle \mathsf{op} \mid C \rangle \to \langle \mathsf{op} \mid \widehat{C} \rangle$, as well as $\Phi_{\widehat{C}}$.

Next, we prove that pointwise left adjoint are left adjoints, where $f :_\flat C \to D$ is a pointwise left adjoint if $\Phi(f^\dagger(-), d)$ is representable for all $d :_\flat D$. Using this we get examples of adjunctions, including the left adjoint $f_!$ to the pullback map $f^* :_\flat \widehat{D} \to \widehat{C}$ induced by a functor $f :_\flat C \to D$.

Finally, let us mention the theory of Kan extensions, where we get the expected results, e.g., if E is a crisp cocomplete category and $f :_\flat C \to D$, then the left Kan extension functor, lan_f, exists, and if $X :_\flat C \to E$, $d : D$, then $\mathsf{lan}_f\ X\, d = \varinjlim(C_{/d} \to C \to E)$.

Prospect of formalization A good part of simplicial type theory has been formalized in Kudasov's proof assistant Rzk [10]. We hope that, pending the development of a user-friendly proof assistant for (a version of) MTT, all of our results should eventually formalizable, too.

References

[1] Daniel Gratzer, Jonathan Weinberger, and Ulrik Buchholtz. *Directed univalence in simplicial homotopy type theory*. 2024. arXiv: 2407.09146 [cs.LO].

[2] Emily Riehl and Michael Shulman. "A type theory for synthetic ∞-categories". In: *Higher Structures* 1 (1 2017), pp. 147–224. DOI: 10.21136/HS.2017.06.

[3] Daniel Gratzer, Jonathan Weinberger, and Ulrik Buchholtz. *The Yoneda embedding in simplicial type theory*. 2025. arXiv: 2501.13229 [cs.LO].

[4] Ulrik Buchholtz and Jonathan Weinberger. "Synthetic fibered $(\infty, 1)$-category theory". In: *Higher Structures* 7 (1 2023), pp. 74–165. DOI: 10.21136/HS.2023.04.

[5] Denis-Charles Cisinski, Bastiaan Cnossen, Kim Nguyen, and Tashi Walde. *Formalization of Higher Categories*. Lecture notes from a course of Denis-Charles Cisinski. 2024. URL: https://drive.google.com/file/d/1lKaq7watGGl3xvjqw9qHjm6SDPFJ2-0o/view.

[6] César Bardomiano Martínez. *Limits and colimits of synthetic ∞-categories*. 2022. arXiv: 2202.12386 [math.CT].

[7] Daniel Gratzer, G.A. Kavvos, Andreas Nuyts, and Lars Birkedal. "Multimodal Dependent Type Theory". In: *Proceedings of the 35th Annual ACM/IEEE Symposium on Logic in Computer Science*. LICS '20. ACM, 2020. DOI: 10.1145/3373718.3394736.

[8] Daniel Gratzer. "Syntax and semantics of modal type theory". PhD thesis. Aarhus University, 2023. URL: https://pure.au.dk/portal/en/publications/syntax-and-semantics-of-modal-type-theory.

[9] Michael Shulman. "Brouwer's fixed-point theorem in real-cohesive homotopy type theory". In: *Mathematical Structures in Computer Science* 28.6 (2018), pp. 856–941. DOI: 10.1017/S0960129517000147.

[10] Nikolai Kudasov, Emily Riehl, and Jonathan Weinberger. "Formalizing the ∞-Categorical Yoneda Lemma". In: *Proceedings of the 13th ACM SIGPLAN International Conference on Certified Programs and Proofs*. 2004, pp. 274–290. DOI: 10.1145/3636501.3636945.