Easy Parametricity

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Parametricity is a wonderful tool for making arguments about definable families of operations simpler: just as one might expect a function on the reals defined in a suitably constructive or choice-free manner to be Lebesgue measurable[1], parametricity witnesses that families of functions $u_X : A(X) \to B(X)$ defined suitably uniformly in the set X should be "nice" and satisfy a corresponding equation such as naturality or dinaturality.

Parametricity typically comes in one of two forms: definable parametricity, which asserts that every family which can be written down in a certain system is parametric; or internal parametricity, which extends a system with additional structure or axioms that allow one to prove that every family is appropriately parametric.

We proceed to take an axiom-based approach to internal parametricity based on asserting a consequence of parametricity in a univalent universe:

Axiom (Parametricity Axiom, version 1)**.** *$\mathcal{U}$ is a univalent universe, and for any type $A : \mathcal{U}$ the map*

$$a \mapsto \lambda\_.a : A \to \prod_{X:\mathcal{U}} A$$

is an equivalence.

Parametricity is usually formulated in a relational guise following Reynolds' relational parametricity [4], but there is precedent for a functor-based formulation of parametricity [3, 8]. The Parametricity Axiom gives strong forms of this:

Theorem 1. *Assume the Parametricity Axiom for $\mathcal{U}$.*

Let $\mathbf{C}$ be a $\mathcal{U}$-complete univalent category and $\mathbf{D}$ be a locally $\mathcal{U}$-small category.

(a) Let $F, G : \mathbf{C} \to \mathbf{D}$ be functors and let $\alpha : \prod_{X:\mathbf{C}} \mathbf{D}(F(X), G(X))$. Then α is natural.

(b) Let $F, G : \mathbf{C}^{\mathsf{op}} \times \mathbf{C} \to \mathbf{D}$ be bifunctors and let $\alpha : \prod_{X:\mathbf{C}} \mathbf{D}(F(X, X), G(X, X))$. Then α is dinatural.

(c) Let $F : \mathbf{C} \to \mathbf{D}$ be a function on objects and morphisms which respects sources, targets and identity morphisms. Then F respects composition, so is a functor.

The axiom holds in useful places, enabling application in practice:

[1] See [6] on what could classify as "suitable" here.

- In cohesive HoTT [9], if the "axiom of sufficient cohesion" or "Axiom C2" holds then the subuniverse of $\int$-modal types satisfies the Parametricity Axiom.
- Analogously, any stably locally connected grothendieck 1-topos satisfying the "axiom of sufficient cohesion" (e.g. cubical sets) has a universe satisfying a suitable non-univalent weakening of the above Parametricity Axiom.
- Whenever $\mathcal{U}$ is an impredicative univalent universe, we show that $\mathcal{U}$ satisfies the Parametricity Axiom:

Theorem 2. *Let $\mathcal{U}$ be an impredicative univalent universe. Then $\mathcal{U}$ satisfies the Parametricity Axiom. Moreover, the "incoherent encoding"*

$$A \mapsto \prod_{X:\mathcal{U}} (A \to X) \to X$$

defines the reflection from arbitrary types to $\mathcal{U}$-small types, and is in fact coherent.

In relation to other results on parametricity This work takes a semantically-inspired approach to parametricity, axiomising the properties required to obtain parametricity results. This contrasts somewhat with the work of Cavallo and Harper [5] and Nuyts, Vezzosi, and Devriese [7], which each augment the syntax. Both enrich dependent type theory with additional judgemental structure (in [5], a type requiring substructural rules; in [7], comonadic modalities) and a notion of "bridge" in order to obtain polymorphism results.

Fully coherent impredicative encodings of higher inductive types have also previously been investigated—Awodey, Frey, and Speight [2] constructed coherent impredicative encodings for several HITs, albeit with elimination rules restricted to types of bounded h-level. Some work to remove these h-level restrictions was made in [10] for some HITs, although a complicated encoding was still required. The present work indicates that plausibly, in the presence of univalence, the naïve encodings for HITs obtain their full induction principles.

This work is closest in spirit to the recent [1], which obtains parametricity results in Cohesive HoTT with sufficient cohesion, also formulated through a bridge-like type. The current work differs from this in two key aspects: the parametricity we consider is functorial instead of relational in nature, and hence a priori more easily scalable to complex structures without having to apply parametricity at each step individually; and no cohesion structure is required (although a $\int$-like modality is helpful), which enables our results to automatically also hold in more general settings of impredicative universes and stably locally connected (∞-)toposes.

Because the current work can be carried out in a traditional type theory without any additional judgmental structure (unlike other works [5, 7, 1]), the standard semantics for MLTT can be immediately used and applied. Further, because this system applies to possibly-non-definable terms of the theory, parametricity results can be applied to functors whose well-definedness relies on extralogical principles such as Countable Choice, as long as such are semantically justified[2].

[2]For example, presheaf topoi over a topos satisfying countable choice (such as **Set**) also satisfy countable choice, so choosing a (sufficiently nontrivial) cohesive presheaf topos over **Set** will also satisfy CC, and hence both CC and Parametricity will be satisfied in the model.

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